On spectral analysis of preconditioners for fully implicit Runge-Kutta systems

Michal Outrata

Outline

- Introduction and Preliminaries
- Preconditioner
- Analysis
- How useful is the analysis anyways?

Model problem

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$$(\partial_{tt} - \Delta)\, u = f \quad \text{in } \Omega \times (0, T)$$

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$$w_t = \begin{bmatrix} & \mathrm{id} \\ \Delta & \end{bmatrix} w + \begin{bmatrix} \\ f \end{bmatrix} \quad \text{in } (0, T) \times \Omega$$

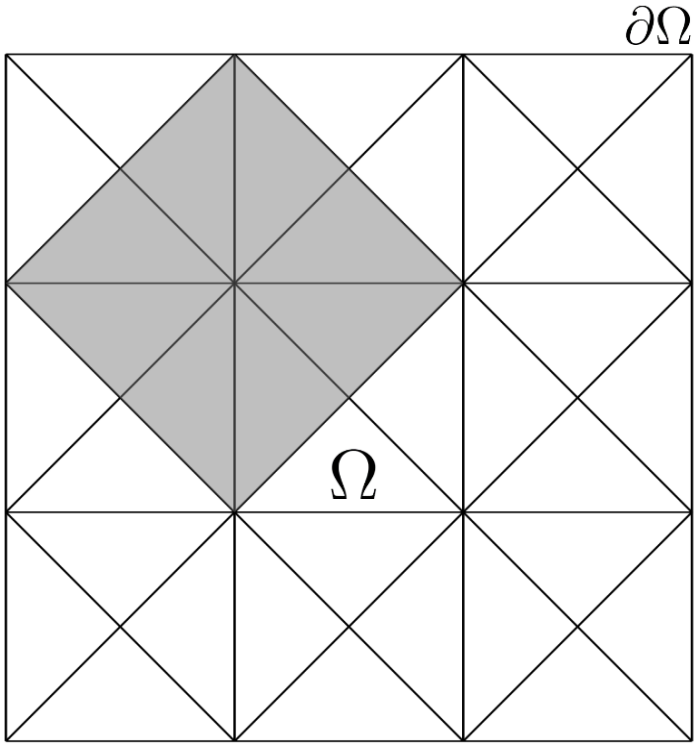
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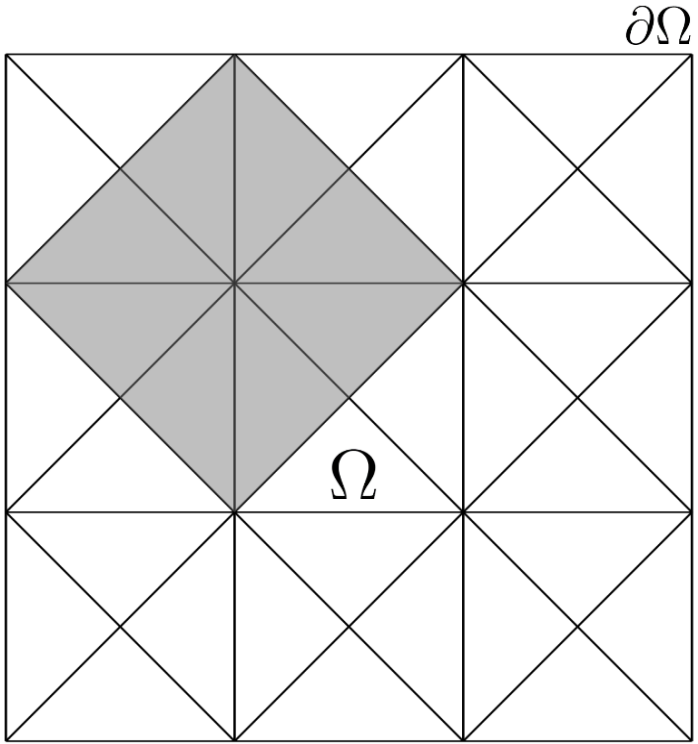
$$w_t = \begin{bmatrix} & \mathrm{id} \\ \Delta & \end{bmatrix} w + \begin{bmatrix} \\ f \end{bmatrix} \quad \text{in } (0, T) \times \Omega$$

$$\text{with} \quad w(t) := \begin{bmatrix} u(\mathbf{x}, t) \\ u_t(\mathbf{x}, t) \end{bmatrix}$$

Model problem

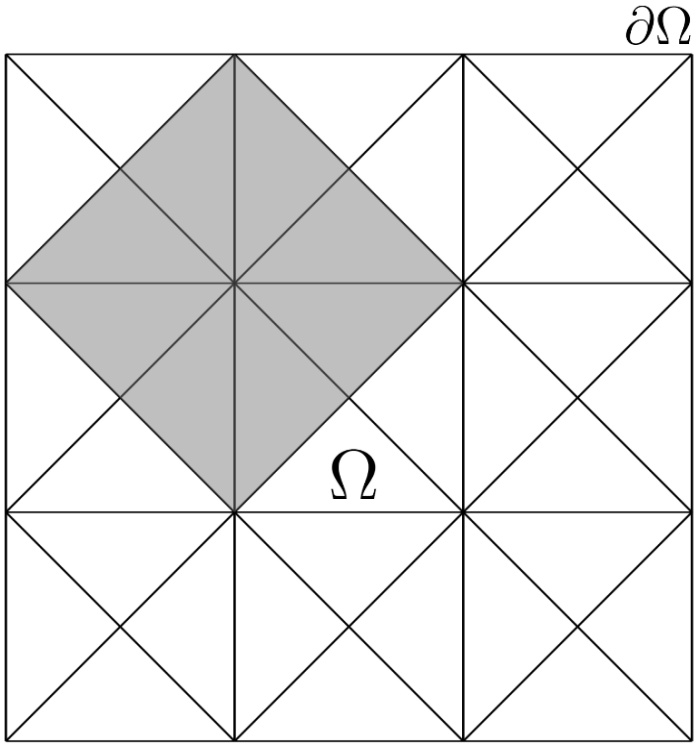


Model problem



$$w(t_m, x) \approx$$
$$\approx \phi_1(x) w_1^{(m)} + \ldots + \phi_n(x) w_n^{(m)}$$

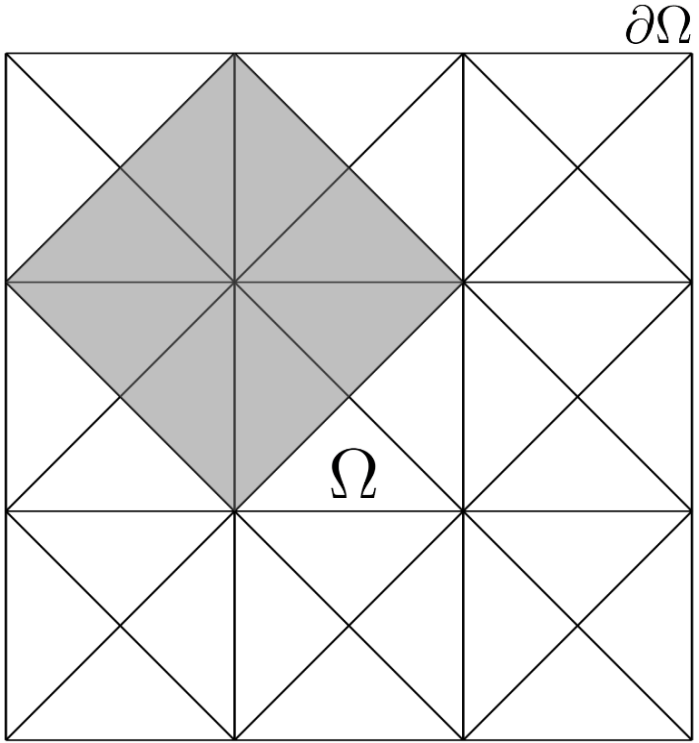
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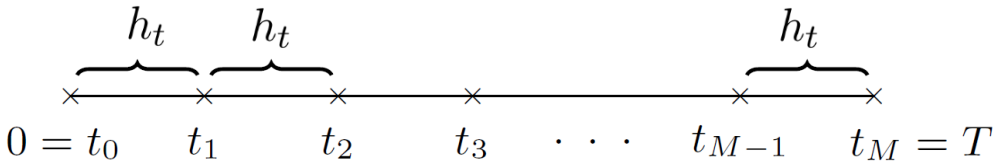
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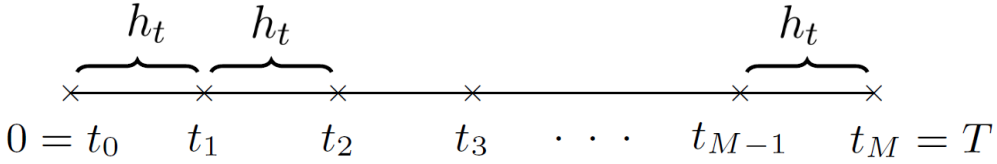
$$\mathcal{M} = \begin{bmatrix} M_\Delta & \\ & M_\Delta \end{bmatrix}, \mathcal{B} = \begin{bmatrix} & I \\ K_\Delta & \end{bmatrix}$$

Runge-Kutta method

Runge-Kutta method

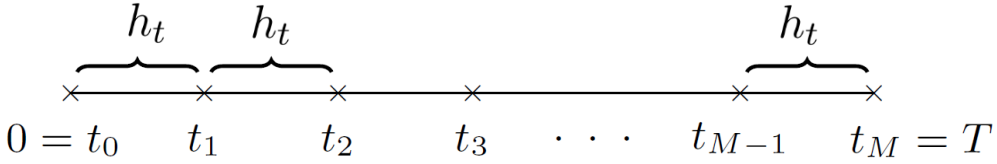


Runge-Kutta method



$$\mathbf{w}^{(m)} = \mathbf{w}^{(m-1)} + h_t \sum_{i=1}^{s} b_i \mathbf{k}_i^{(m)}$$

Runge-Kutta method



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$$\left(A^{-1} \otimes I_{2N} + h_t I_s \otimes \mathcal{M}^{-1} \mathcal{B} \right) \hat{\mathbf{k}}^{(m)} = \hat{\mathbf{b}}_{\mathrm{IRK}}^{(m)}$$

Reformulaiton

$$\left(A^{-1} \otimes \mathcal{I}_{2N} + h_t I_s \otimes \mathcal{M}^{-1}\mathcal{B}\right) \hat{\mathbf{k}}^{(m)} = \hat{\mathbf{b}}^{(\mathrm{IRK})}$$

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Aman Rani

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Aman Rani

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$$\mathcal{M}\bar{K}^{(m)}D - h_t\mathcal{B}\bar{K}^{(m)} = \bar{B}^{(\mathrm{IRK})}$$

$$\left(\mathcal{M} - \frac{h_t}{d_i}\mathcal{B}\right) \bar{\mathbf{k}}_i^{(m)} = \bar{\mathbf{b}}_i^{(\mathrm{IRK})}, \quad i = 1, \ldots, s$$

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Aman Rani

Ivo Dravins

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IRK linear systems

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$$s \to \left\lceil \frac{s}{2} \right\rceil \;\&\; \mathcal{T}_i \to \mathcal{P}^{-1}\mathcal{T}_i \;\&\; \mathcal{P}^{-1} \in \mathbb{R}^{2N \times 2N}$$

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$$\mathcal{T}_i = \mathcal{M} - \frac{h_t}{d_i}\mathcal{B} = \begin{bmatrix} M_\Delta & 0 \\ \frac{h_t}{d_i}K_\Delta & M_\Delta + \left(\frac{h_t}{d_i}\right)^2 K_\Delta \end{bmatrix} \begin{bmatrix} I & -\frac{h_t}{d_i}I \\ & I \end{bmatrix}$$

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$$\begin{bmatrix} M_\Delta & 0 \\ \frac{h_t}{d_i}K_\Delta & M_\Delta + \left(\frac{h_t}{d_{\text{avg}}}\right)^2 K_\Delta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & -\frac{h_t}{d_i}I \\ & I \end{bmatrix}$$

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$$\mathcal{P}_i \quad \text{“} = \text{”} \quad \begin{bmatrix} M_\Delta & -\frac{h_t}{d_i} M_\Delta \\ \frac{h_t}{d_i} K_\Delta & M_\Delta + h_t^2 \frac{d_i^2 - d_{\text{avg}}^2}{d_{\text{avg}}^2 d_i^2} K_\Delta \end{bmatrix}$$

Preconditioner analysis

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Assume $\{M_\Delta, K_\Delta\}$ is diagonalizable

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$$\mathcal{V} := \begin{bmatrix} V & \\ & V \end{bmatrix} \quad \text{and} \quad \mathcal{W} := \begin{bmatrix} W & \\ & W \end{bmatrix}$$

Preconditioner analysis

$$\mathcal{V}^T \begin{bmatrix} \beta_{11} M_\Delta + \gamma_{11} K_\Delta & \beta_{12} M_\Delta + \gamma_{12} K_\Delta \\ \beta_{21} M_\Delta + \gamma_{21} K_\Delta & \beta_{22} M_\Delta + \gamma_{22} K_\Delta \end{bmatrix} \mathcal{W}$$

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$$= \begin{bmatrix} \beta_{11} \Lambda_M + \gamma_{11} \Lambda_K & \beta_{12} \Lambda_M + \gamma_{12} \Lambda_K \\ \beta_{21} \Lambda_M + \gamma_{21} \Lambda_K & \beta_{22} \Lambda_M + \gamma_{22} \Lambda_K \end{bmatrix}$$

Preconditioner analysis

$$\mathcal{P}_i^{-1}\mathcal{T}_i$$

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Preconditioner analysis

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Preconditioner analysis

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \sim \begin{bmatrix} \zeta_k^{(11)} & \zeta_k^{(12)} \\ \zeta_k^{(21)} & \zeta_k^{(22)} \end{bmatrix}$$

with $Z_{ij} = \mathrm{diag}\left(\zeta_1^{(ij)}, \ldots, \zeta_N^{(ij)}\right)$

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$$\begin{bmatrix} \zeta^{(11)} & \zeta^{(12)} \\ \zeta^{(21)} & \zeta^{(22)} \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda_M & -\frac{h_t}{d_i}\lambda_M \\ \frac{h_t}{d_i}\lambda_K & \lambda_M + h_t^2 \frac{d_i^2 - d_{\text{avg}}^2}{d_{\text{avg}}^2 d_i^2}\lambda_K \end{bmatrix}^{-1} \begin{bmatrix} \lambda_M & -\frac{h_t}{d_i} \\ -\frac{h_t}{d_i}\lambda_K & \lambda_M \end{bmatrix}$$

Preconditioner analysis

Lemma.

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Lemma. *Let* λ_M, λ_K *and* $\mathbf{v}, \mathbf{w}$ *be the eigenvalues and eigenvectors of the pencil* $\{M_\Delta, K_\Delta\}$ *as above.*

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$$\left(1, \Pi \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \mathbf{w}\right) \quad \text{and} \quad \left(\xi_i, \Pi \begin{bmatrix} \eta_i \\ -\eta_i \end{bmatrix} \otimes \mathbf{w}\right)$$

Preconditioner analysis

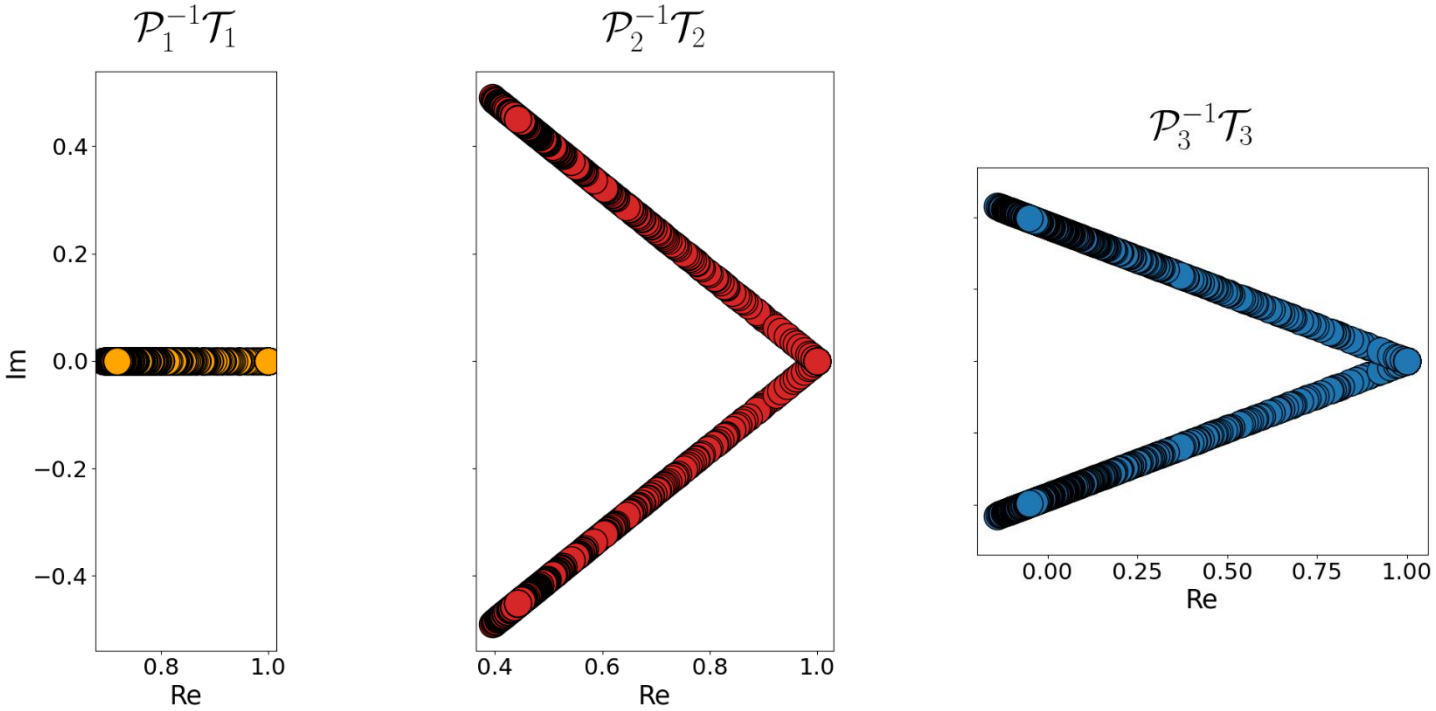
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with $\theta := h_t^2 \lambda_K / \lambda_M$ and

$$\xi_i := \frac{1 + \theta/d_i^2}{1 + \theta/d_{\text{avg}}^2} \quad \text{and} \quad \eta_i := \frac{1}{\sqrt{1 + |\gamma^i|^2}} \;\&\; |\gamma^i|^2 = \frac{|d_i|^2/h_t^2}{|1 + \theta/d_{\text{avg}}^2|^2}$$

Preconditioner analysis



Preconditioner analysis

Proposition.

Preconditioner analysis

Proposition. *Assume that the eigenvalues* λ_M, λ_K *are real and let* ξ_1, ξ_2 *be eigenvalues of the* i*-th preconditioned system* $\mathcal{P}_i^{-1}\mathcal{T}_i$.

Preconditioner analysis

Proposition. *Assume that the eigenvalues λ_M, λ_K are real and let ξ_1, ξ_2 be eigenvalues of the i-th preconditioned system $\mathcal{P}_i^{-1}\mathcal{T}_i$. Then*

$$\frac{\mathrm{Im}\,(\xi_1) - \mathrm{Im}\,(\xi_2)}{\mathrm{Re}\,(\xi_1) - \mathrm{Re}\,(\xi_2)} = \frac{\mathrm{Im}\,(1/d_i^2)}{\mathrm{Re}\,(1/d_i^2) + d_{\mathrm{avg}}}.$$

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In words, the eigenvalues of the preconditioned system lie on a line in $\mathbb{C}$.

Preconditioner analysis

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$$\kappa\left(\mathcal{S}_i\right) \leq \kappa\left(W\right) \cdot \sqrt{\frac{\sqrt{1+\min_\theta |\gamma_\theta^i|^2}+1}{\sqrt{1+\min_\theta |\gamma_\theta^i|^2}-1}},$$

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with

$$\theta := h_t^2 \lambda_K / \lambda_M \quad \text{and} \quad |\gamma_\theta^i|^2 = \frac{|d_i|^2/h_t^2}{|1+\theta/d_{\text{avg}}^2|^2}.$$

Preconditioner analysis

sp.linalg.gmres

Preconditioner analysis

`sp.linalg.gmres`

$$\frac{\|\mathbf{r}_\ell\|}{\|\mathbf{r}_0\|} \leq \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq\ell}} \|\varphi(\mathcal{P}_i^{-1}\mathcal{T}_i)\|$$

$$\frac{\|\mathbf{r}_\ell\|}{\|\mathbf{r}_0\|} \leq \kappa(\mathcal{S}_i) \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq\ell}} \max_{\xi\in\mathrm{sp}(\mathcal{P}_i^{-1}\mathcal{T}_i)} |\varphi(\xi)|$$

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$$\frac{\|\mathbf{r}_\ell\|}{\|\mathbf{r}_0\|} \leq \kappa(\mathcal{S}_i) \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq\ell}} \max_{\xi\in\left[\xi_1^{(i)},\xi_N^{(i)}\right]} |\varphi(\xi)|$$

Numerical example

Numerical example

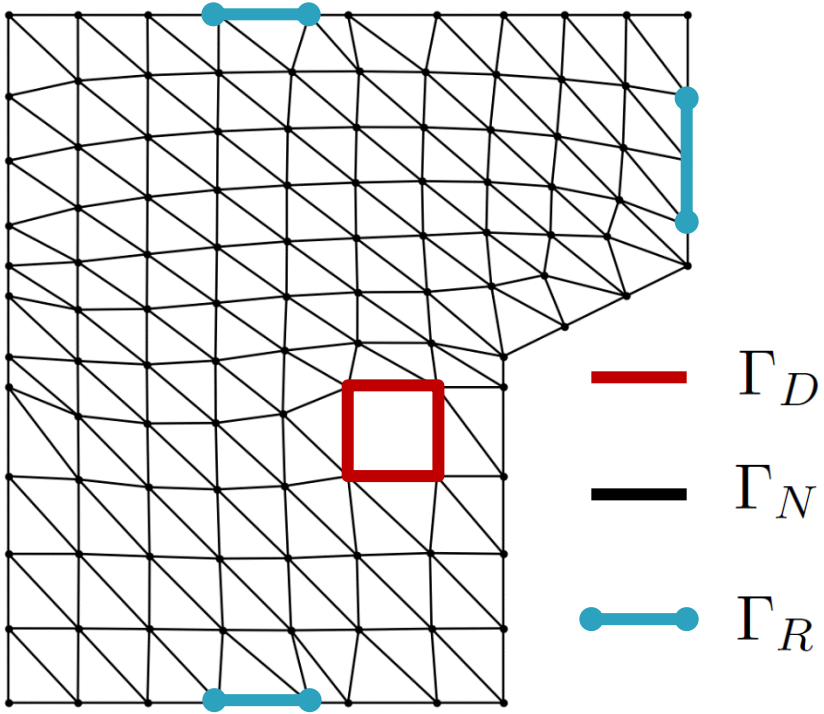
$$(\partial_{tt} - \Delta)\, u = \Delta u \quad \text{in } \Omega \times (0, T]$$

$$\frac{\partial u}{\partial \mathbf{n}} u = 0 \quad \text{on } \Gamma_N \times (0, T],$$

$$\frac{\partial u}{\partial \mathbf{n}} u + pu = 0 \quad \text{on } \Gamma_R \times (0, T],$$

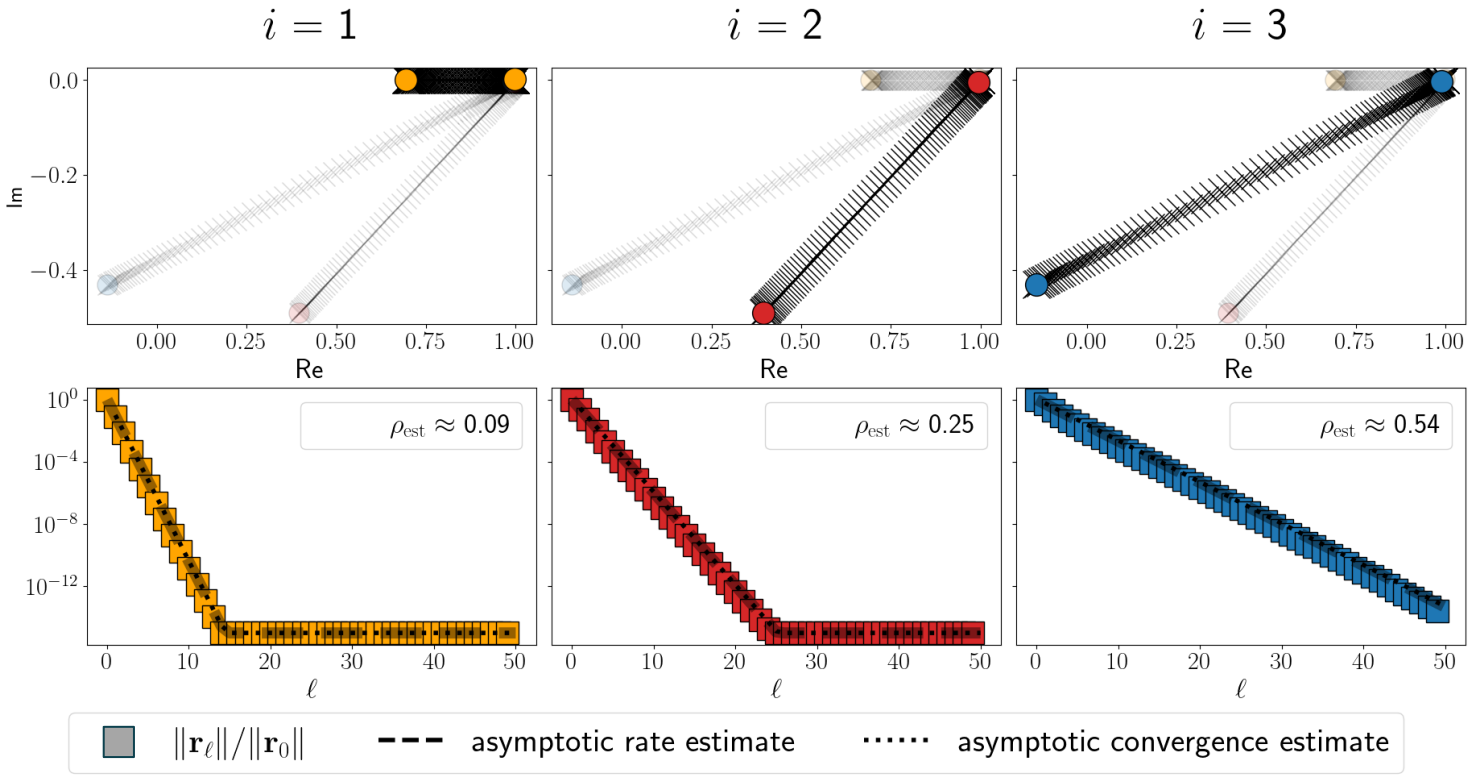
$$u = 0 \quad \text{at } \Omega \times \{0\}$$

Numerical example





Numerical example



Conclusion

Thank you for your attention

IRK preconditioners

IRK preconditioners

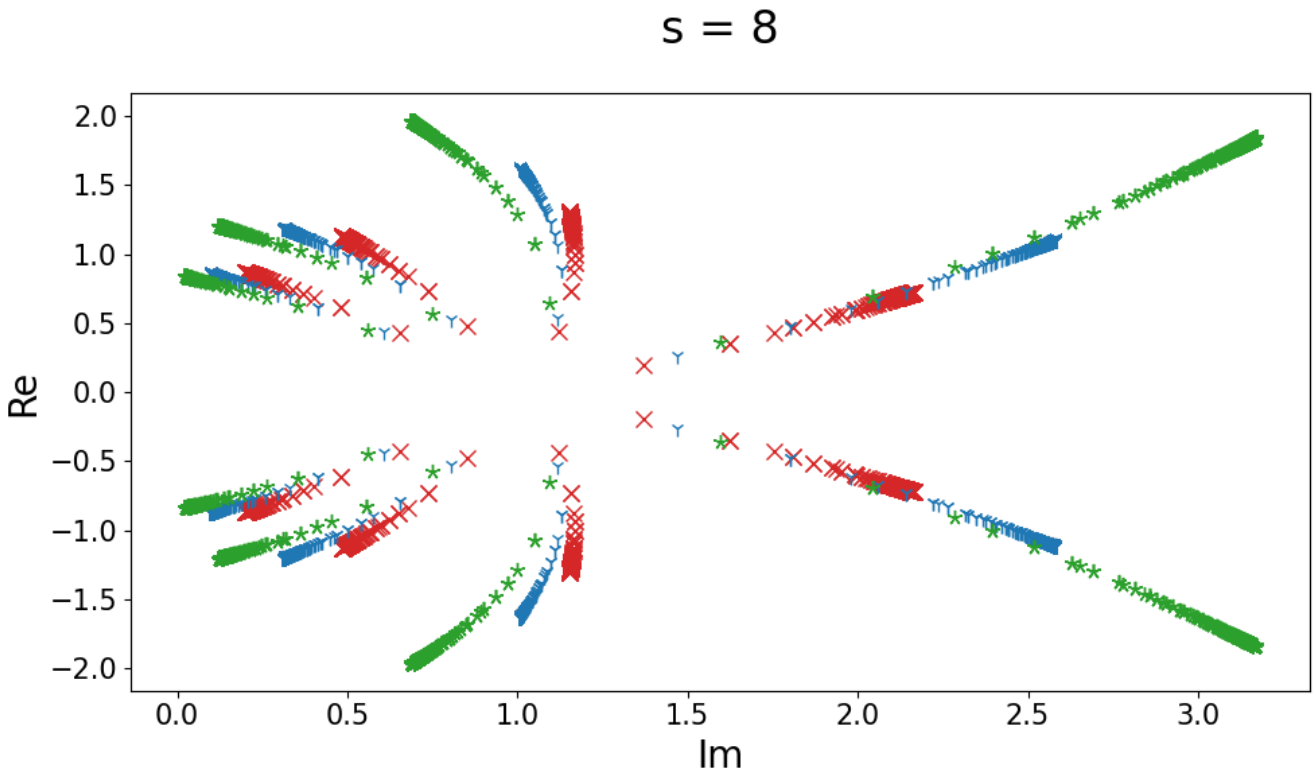
IRK preconditioners

How about

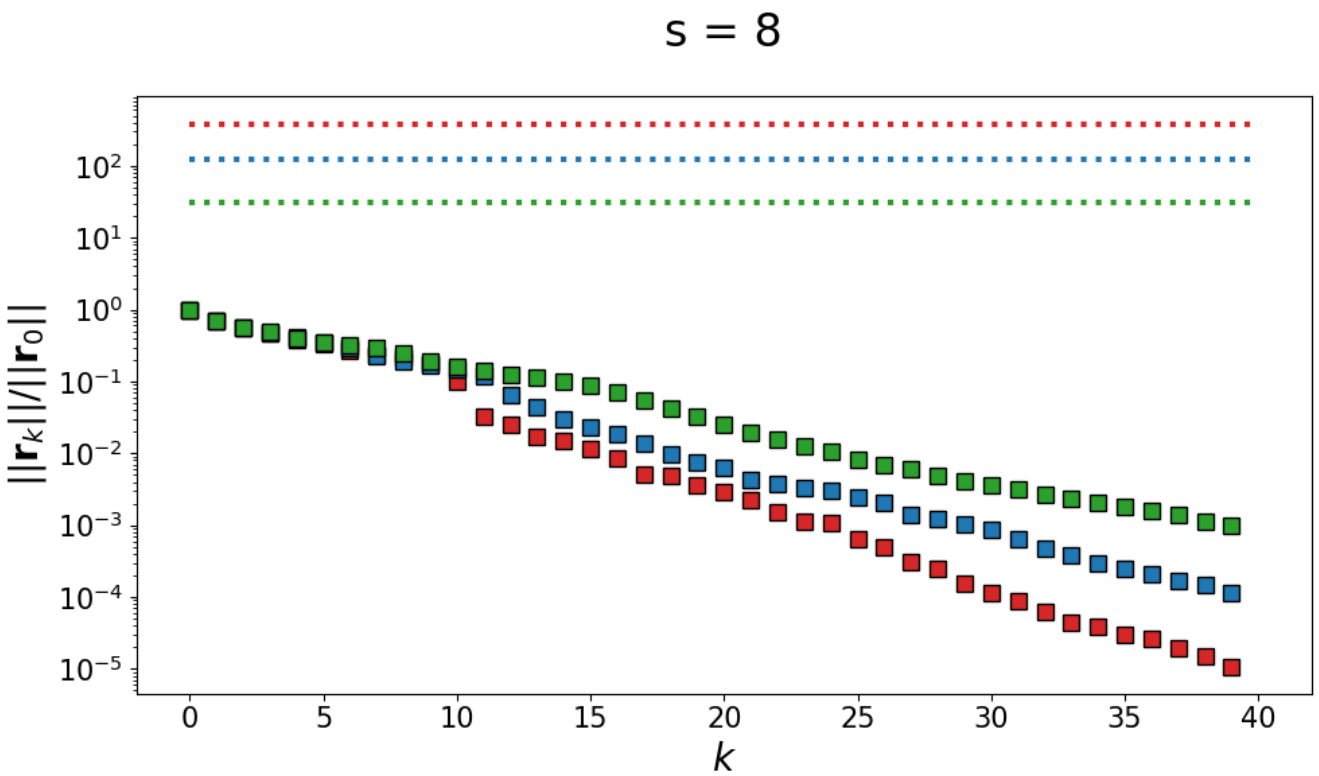
$$s \geq 2$$



IRK preconditioners



IRK preconditioners



Preconditioner analysis

`sp.linalg.gmres`

$$\frac{\|r_k\|}{\|r_0\|} \leq \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \|\varphi(\, M\left(P^{\text{diag}}\right)^{-1}\,)\|$$

$$\frac{\|r_k\|}{\|r_0\|} \leq \kappa(S) \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \max_{\zeta_i \in \text{sp}(\, M\left(P^{\text{diag}}\right)^{-1}\,)} |\varphi(\zeta_i)|$$

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$$\frac{\|r_k\|}{\|r_0\|} \leq \kappa(S) \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \max_{\zeta \in \sigma} |\varphi(\zeta)|$$

Is our σ functional?

$$\frac{\|r_k\|}{\|r_0\|} \le \kappa(S) \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\le k}} \max_{\zeta \in \sigma} |\varphi(\zeta)|$$

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FROM POTENTIAL THEORY TO MATRIX ITERATIONS IN SIX STEPS*

TOBIN A. DRISCOLL†, KIM-CHUAN TOH‡, AND LLOYD N. TREFETHEN§

To the memory of Peter Henrici

Choosing σ

Choosing σ

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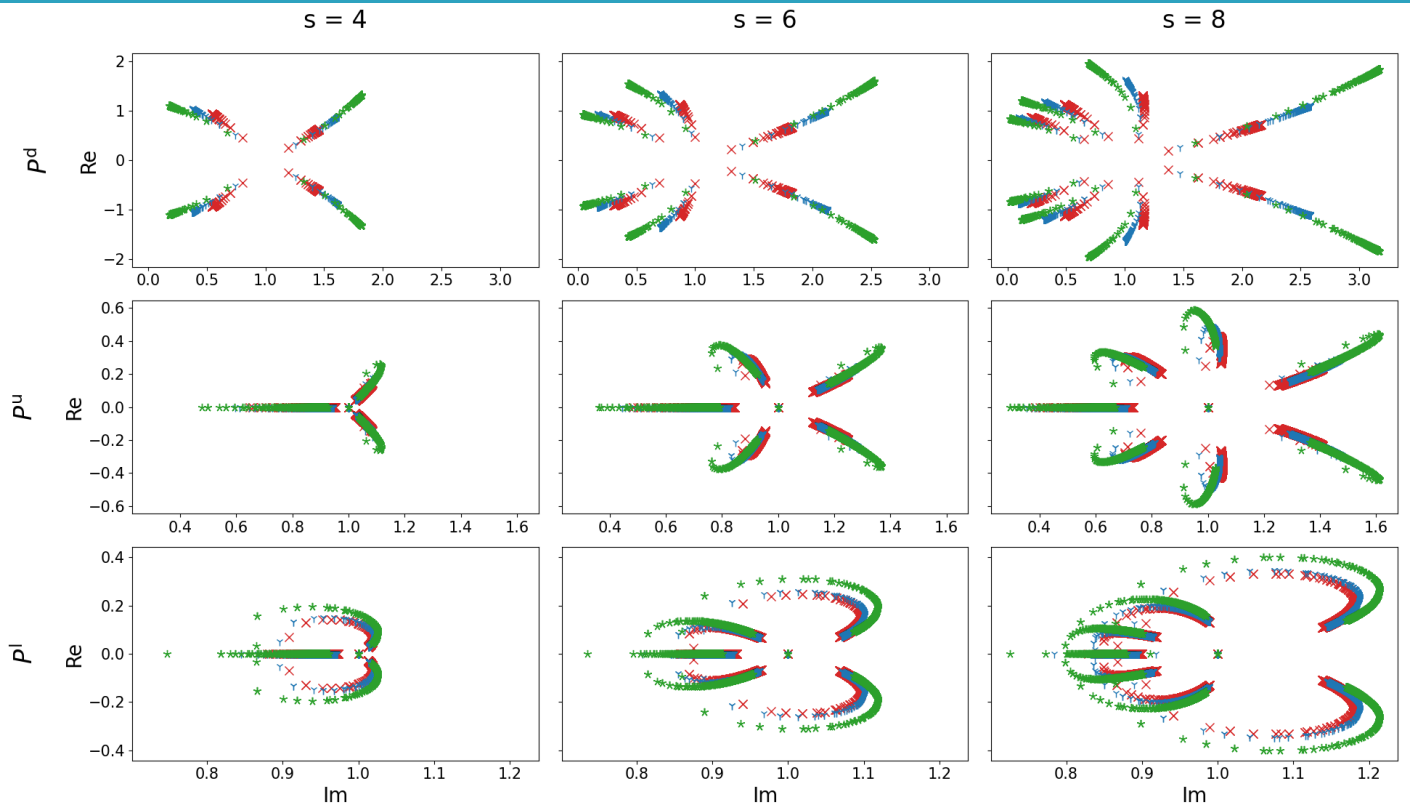
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Choosing σ should be

- *appropriate*: $\sigma \approx \partial_{\mathbb{C}} \operatorname{sp}\left(M(P^{\text{diag}})^{-1}\right)$
- *functional*: bound or evaluate the min-max over σ

Choosing σ appropriately



Is our σ functional?

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FROM POTENTIAL THEORY TO MATRIX ITERATIONS IN SIX STEPS*

TOBIN A. DRISCOLL†, KIM-CHUAN TOH‡, AND LLOYD N. TREFETHEN§

To the memory of Peter Henrici

If σ is compact and connected, then it's functional.

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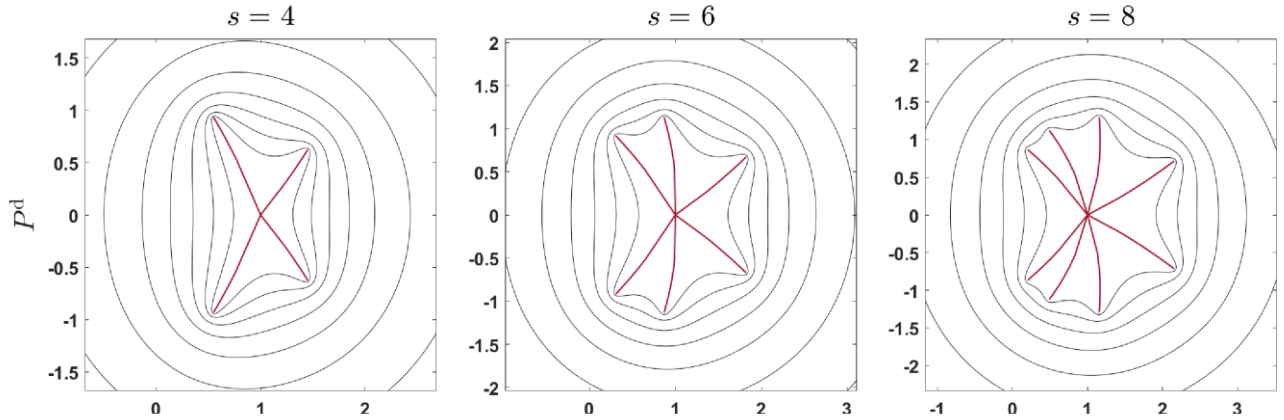
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Numerical example

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$$\frac{\partial u}{\partial t} = \mathbf{a} \cdot \nabla u + f \quad \text{in } \Omega \times (0, T],$$

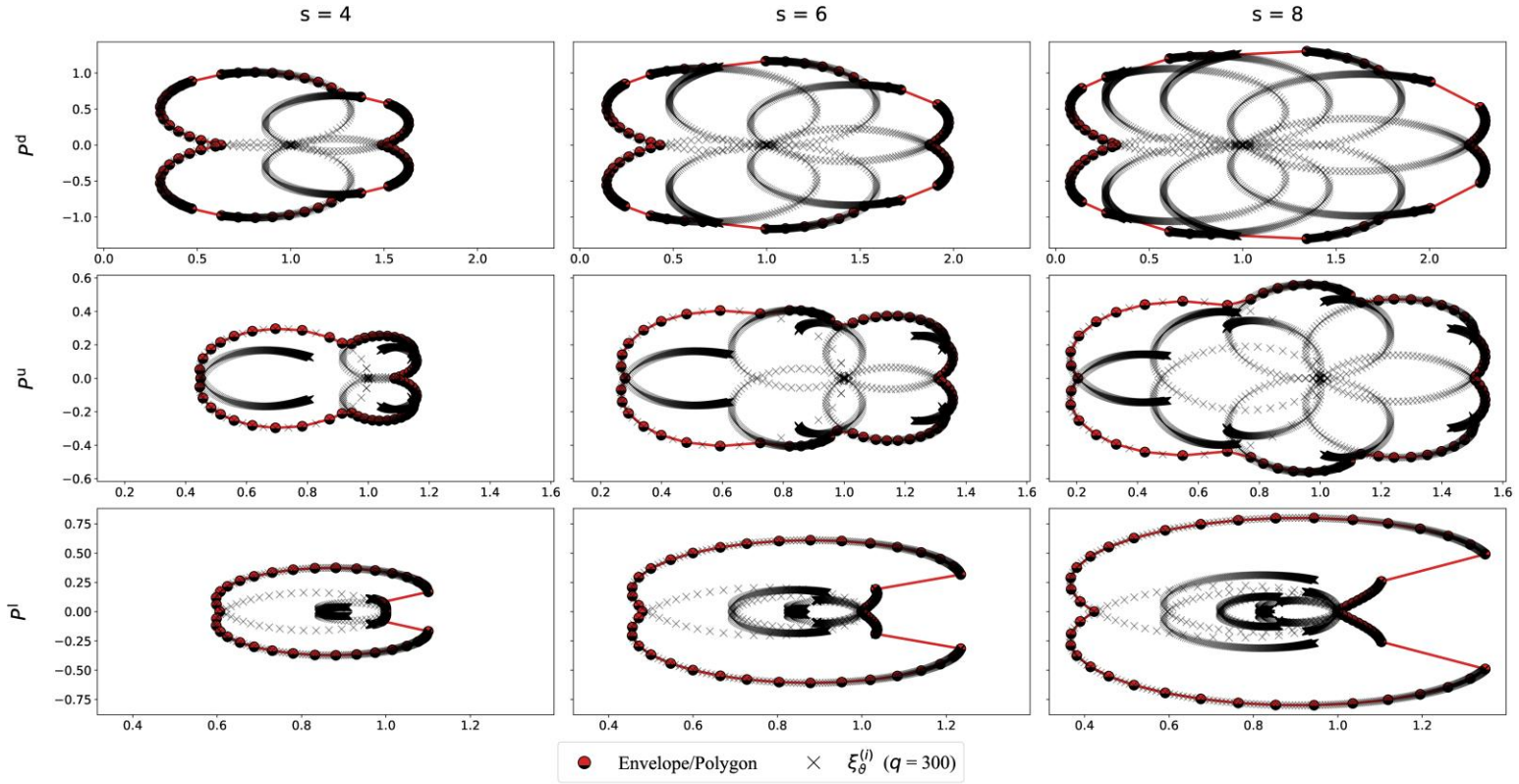
$$u = 0 \quad \text{on } \partial\Omega \times (0, T],$$

$$u = 0 \quad \text{at } \Omega \times \{0\}$$

Numerical example

The choice of σ

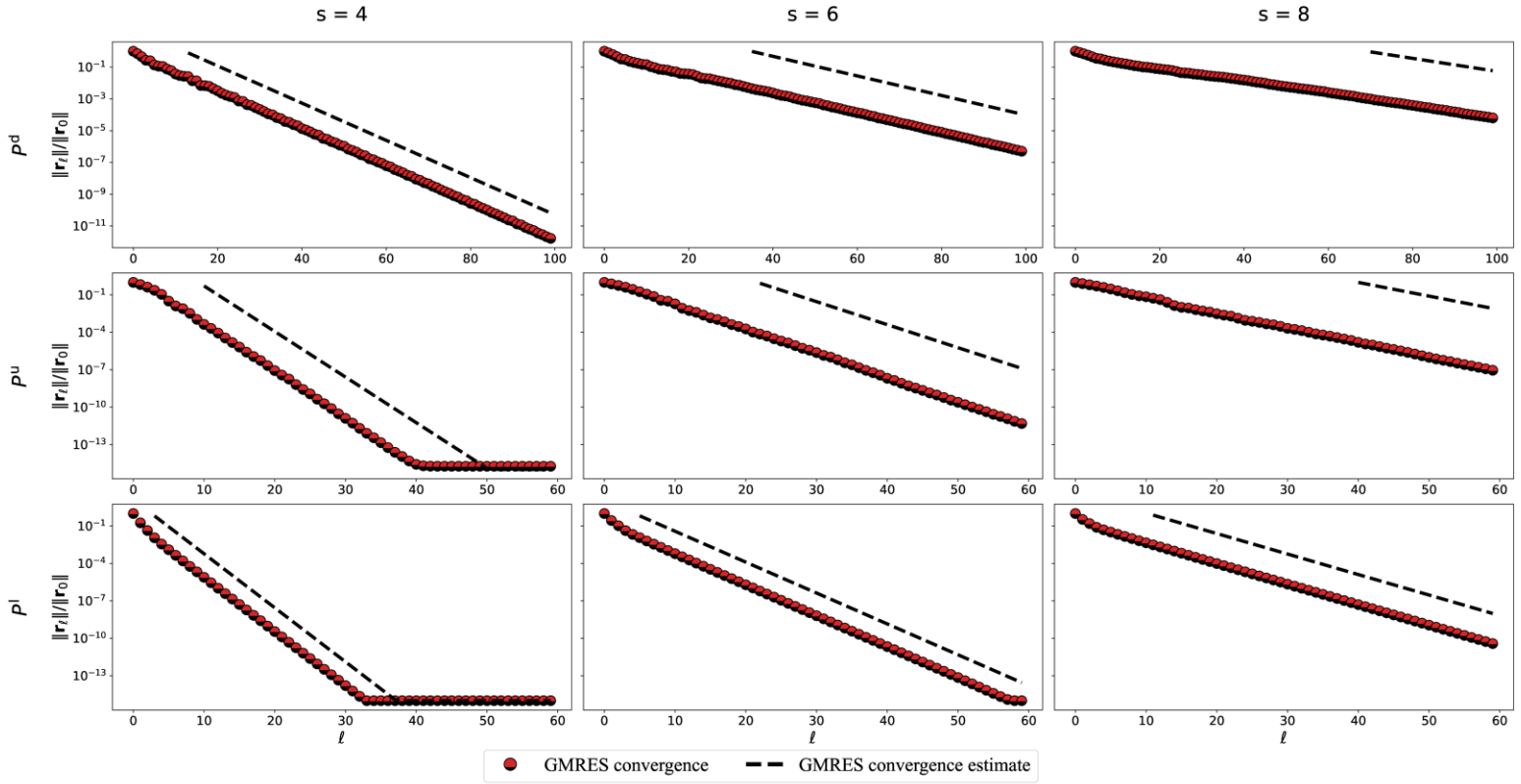
Numerical example



Numerical example

The average GMRES performance

Numerical example



Numerical example

Numerical example

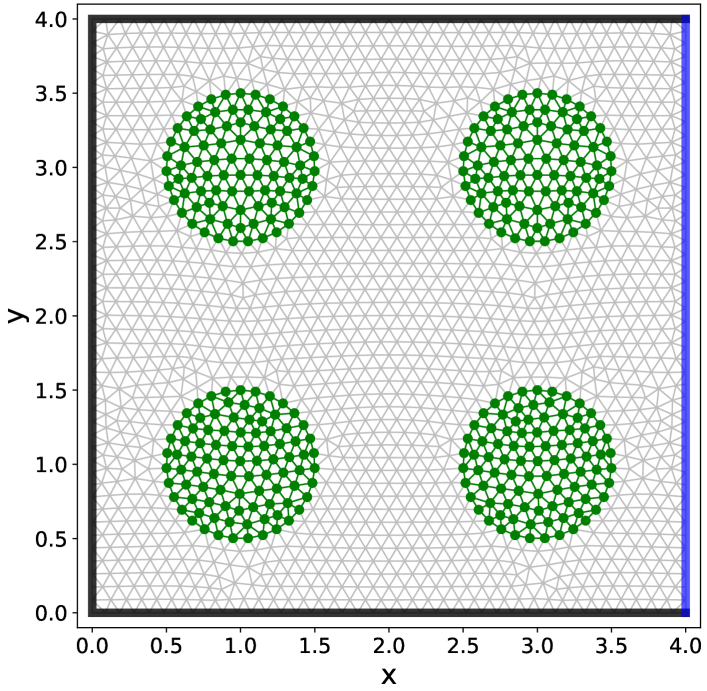
$$\frac{\partial u}{\partial t} u = \operatorname{div}(\nu \nabla u) \quad \text{in } \Omega \times (0, T],$$

$$\frac{\partial u}{\partial \mathbf{n}} u = 0 \quad \text{on } \Gamma_N \times (0, T],$$

$$\frac{\partial u}{\partial \mathbf{n}} u + pu = 0 \quad \text{on } \Gamma_R \times (0, T],$$

$$u = 0 \quad \text{at } \Omega \times \{0\}$$

Numerical example



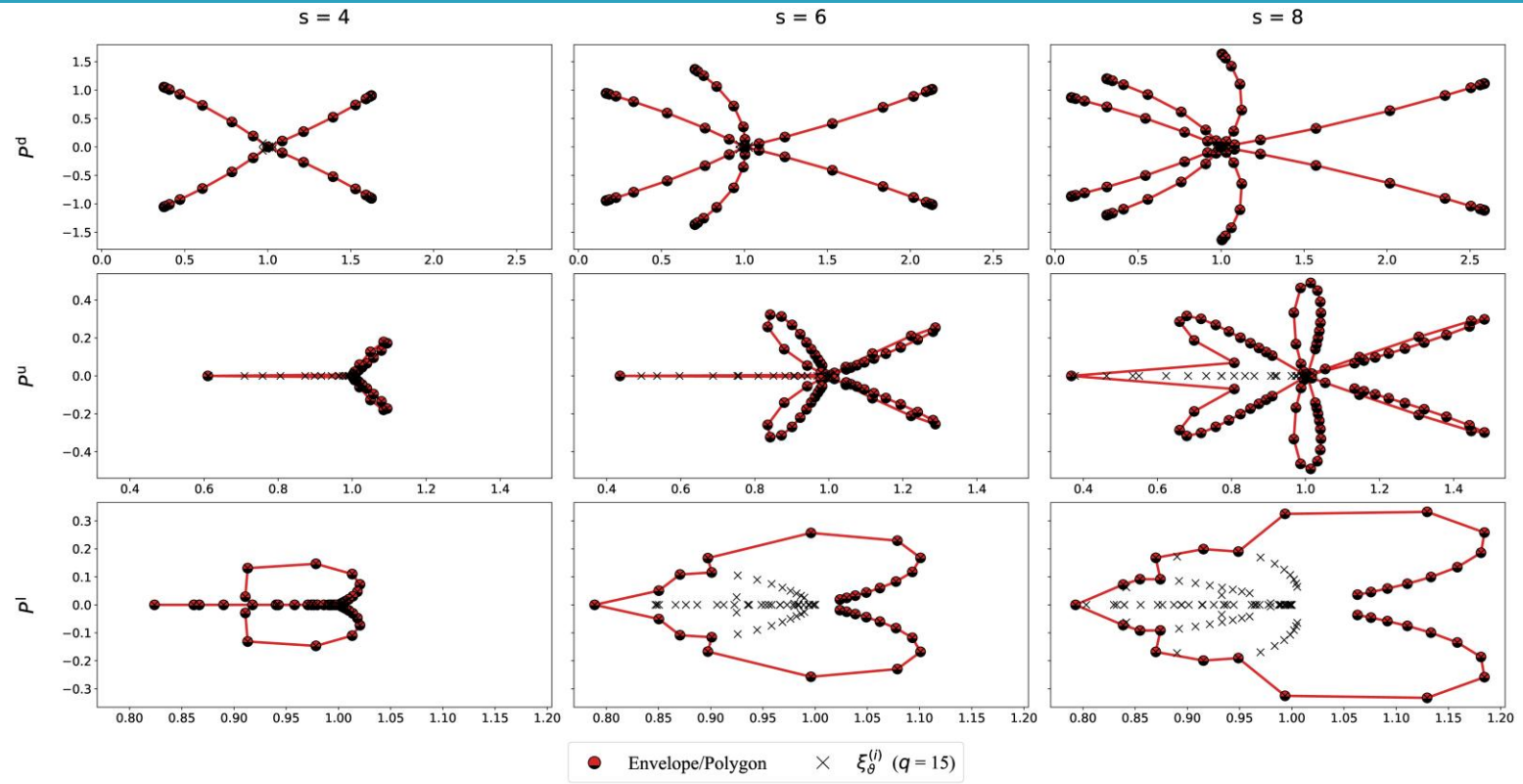
$$\nu = \begin{cases} 10^3 & \text{if } (x, y) \in \text{Cookie}, \\ 1 & \text{otherwise}, \end{cases}$$

Cookies | Dirichlet boundary | Neumann boundary | Robin boundary

Numerical example

The choice of σ

Numerical example



Numerical example

The average GMRES performance

Numerical example

