On direct multilevel solvers induced by AMG

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Set-up

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$$L\mathbf{u} = \mathbf{f}$$

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$L \in \mathbb{C}^{N\times N}$ & $\mathbf{f} \in \mathbb{C}^{N}$

Multigrid background

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Algorithm 1 Two-level AMG $\mathbf{v} \mapsto M^{-1}\mathbf{v}$

1:
2:
3:
4:
5:
6:
7:
8:
9:

Multigrid background

Algorithm 1 Two-level AMG $\mathbf{v} \mapsto M^{-1}\mathbf{v}$

$\mathbf{x} \leftarrow \mathbf{0}$

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x ← 0
for i = 1, ..., m do
    x ← x + α_i S^{-1}(v − Lx)
end for
```

$$\text{pre} - \text{smoothing}$$

$$\alpha_1, \ldots, \alpha_m\,,\ S^{-1}$$

Multigrid background

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x ← x + P M_0^{-1} R(v − Lx)
```

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for $i = 1, \ldots, m$ **do**

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pre − smoothing

$\alpha_1, \ldots, \alpha_m, \; S^{-1}$

coarse correction

$P, \; R, \; M_0^{-1}$

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$$\alpha_1, \ldots, \alpha_m\,, \ S^{-1}$$

$$\text{coarse correction}$$
$$P, \ R\,, \ M_0^{-1}$$

$$\text{post} - \text{smoothing}$$
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Existing direct method

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LDU factorization of L^{-1}

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$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}B \\ & I \end{bmatrix} \begin{bmatrix} A^{-1} & \\ & K^{-1} \end{bmatrix} \begin{bmatrix} I & \\ -CA^{-1} & I \end{bmatrix}$$

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$$K = D - CA^{-1}B$$

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$$S^{-1} = \begin{bmatrix} A^{-1} & \\ & \end{bmatrix} \quad P = \begin{bmatrix} -A^{-1}B \\ I \end{bmatrix} \quad R = \begin{bmatrix} \star & I \end{bmatrix} \quad M_0^{-1} = (RLP)^{-1} = K$$

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T. A. Manteuffel, S. Münzenmaier, J. Ruge, and B. Southworth.
Nonsymmetric reduction-based algebraic multigrid.
Journal on Scientific Computing, 2019.

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Why direct methods?

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$$\boxed{\text{symmetric/palindromic structure of the cycle}}$$

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direct?

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Error propagation:

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$$\boxed{E^{(s)}(\alpha_i) = I - \alpha_i S^{-1} L}$$

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Error propagation

$$E(\alpha_1, \ldots, \alpha_m) =$$

$$\left(\prod_{i=m}^{1} E^{(s)}(\alpha_i) \right) E^{(c)} \left(\prod_{i=1}^{m} E^{(s)}(\alpha_i) \right)$$

Spectral analysis

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$$E^{(s)}\tilde{V} = \begin{bmatrix} (1-\alpha)I & -\alpha A^{-1}B \\ -\alpha D^{-1}C & (1-\alpha)I \end{bmatrix} \begin{bmatrix} W\sqrt{\Lambda} & W\sqrt{\Lambda} \\ D^{-1}CW & -D^{-1}CW \end{bmatrix}$$

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$$E^{(s)}\tilde{V} = \begin{bmatrix} W\sqrt{\Lambda} & W\sqrt{\Lambda} \\ D^{-1}CW & -D^{-1}CW \end{bmatrix} \begin{bmatrix} (1-\alpha-\alpha\sqrt{\Lambda})I & \\ & (1-\alpha+\alpha\sqrt{\Lambda})I \end{bmatrix}$$

Spectral analysis

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$$E^{(s)} \sim \begin{bmatrix} (1-\alpha-\alpha\sqrt{\Lambda})I & \\ & (1-\alpha+\alpha\sqrt{\Lambda})I \end{bmatrix}$$

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$$E^{(c)}\tilde{V} = \begin{bmatrix} I & A^{-1}B \\ & \end{bmatrix} \begin{bmatrix} W\sqrt{\Lambda} & W\sqrt{\Lambda} \\ D^{-1}CW & -D^{-1}CW \end{bmatrix}$$

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$$E^{(c)}\tilde{V} = \begin{bmatrix} W\sqrt{\Lambda} & W\sqrt{\Lambda} \\ D^{-1}CW & -D^{-1}CW \end{bmatrix} \begin{bmatrix} \frac{1}{2}(I + \sqrt{\Lambda}) & \frac{1}{2}(I - \sqrt{\Lambda}) \\ \frac{1}{2}(I + \sqrt{\Lambda}) & \frac{1}{2}(I - \sqrt{\Lambda}) \end{bmatrix}$$

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Spectral analysis

$$E(\alpha_1, \ldots, \alpha_m) = \left(\prod_{i=m}^{1} E^{(s)}(\alpha_i) \right) E^{(c)} \left(\prod_{i=1}^{m} E^{(s)}(\alpha_i) \right)$$

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$$\boxed{\prod_{i=1}^{m} \begin{bmatrix} (1 - \alpha_i - \alpha_i \sqrt{\lambda_k}) & \\ & (1 - \alpha_i + \alpha_i \sqrt{\lambda_k}) \end{bmatrix}}$$

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eigenproperties of $E(\alpha_1, \dots, \alpha_m)$ given by matrices in $\mathbb{C}^{2\times 2}$

Spectral analysis

$$T := A^{-1}BD^{-1}C$$

Assumptions

T is diagonalizable

Spectral analysis

Proposition. *If* $(\lambda, \mathbf{w})$ *is an eigenpair of* T *with* $\lambda \neq 0$*, then*

Spectral analysis

Proposition. *If* $(\lambda, \mathbf{w})$ *is an eigenpair of* T *with* $\lambda \neq 0$*, then*

$$\left(0, \begin{bmatrix} \left(\rho_+(1-\sqrt{\lambda}) - \rho_-(1+\sqrt{\lambda})\right)\sqrt{\lambda}\mathbf{w} \\ \left(\rho_+(1-\sqrt{\lambda}) + \rho_-(1+\sqrt{\lambda})\right) D^{-1}C\mathbf{w} \end{bmatrix}\right), \quad \left(\rho_\lambda, \frac{1}{2}\begin{bmatrix} (\rho_- + \rho_+)\sqrt{\lambda}\mathbf{w} \\ (\rho_- - \rho_+)D^{-1}C\mathbf{w} \end{bmatrix}\right)$$

are eigenpairs of $E(\alpha_1, \ldots, \alpha_m)$ *with*

$$\rho_\lambda := \frac{1}{2}\left((1+\sqrt{\lambda})\rho_-^2 + (1-\sqrt{\lambda})\rho_+^2\right) \quad \text{and} \quad \rho_\pm := \prod_{i=1}^{m}(1-\alpha_i \pm \alpha_i\sqrt{\lambda}).$$

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$$\left(0, \begin{bmatrix} 0 \\ D^{-1}C\mathbf{z} \end{bmatrix}\right), \quad \left(\prod_{i=1}^{m}(1-\alpha_i)^2, \begin{bmatrix} \prod_{i=1}^{m}(1-\alpha_i)\mathbf{z} \\ \sum_{i=1}^{m}(-\alpha_i)\prod_{j\neq i}(1-\alpha_j)D^{-1}C\mathbf{z} \end{bmatrix}\right),$$

are eigenpairs of $E(\alpha_1, \ldots, \alpha_m)$ *with*

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$$m=1: \quad \rho_\lambda(\alpha) = (1-\alpha)^2 + \alpha(3\alpha-2)\lambda$$

Direct ?

Proposition. *Let* $m \geq 1$ *and for* $i = 1, \ldots, m$ *set*

$$c_i = -\cos\left(\frac{2\pi i}{2m+1}\right) \quad \text{and} \quad \hat{\alpha}_i := \frac{1}{1+c_i}.$$

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Hence, the spectrum of $E(\alpha_1, \ldots, \alpha_m)$ *is clustered in two points,*

$$\sigma\left(E(\alpha_1, \ldots, \alpha_m)\right) = \left\{0, \frac{1}{(2m+1)^2}\right\}.$$

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$$\left(M^{-1}L - I\right)\left(M^{-1}L - \frac{(2m+1)^2 - 1}{(2m+1)^2} I\right) = 0$$

The direct method

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$$L^{-1} = \left(\left(1 + \frac{(2m+1)^2}{(2m+1)^2 - 1}\right) I - \frac{(2m+1)^2}{(2m+1)^2 - 1} M^{-1}L\right) M^{-1}$$

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$$\text{two applications} \longrightarrow \text{direct}$$

$$\text{two iterations of CG/GMRES} \longrightarrow \text{direct}$$

The multivelel direct method

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Algorithm 1 Two-level AMG $\mathbf{v} \mapsto M^{-1}\mathbf{v}$

$\mathbf{x} \leftarrow \mathbf{0}$
for $i = 1, \ldots, m$ **do**
    $\mathbf{x} \leftarrow \mathbf{x} + \alpha_i S^{-1}(\mathbf{v} - L\mathbf{x})$
end for
$\mathbf{x} \leftarrow \mathbf{x} + P M_0^{-1} R(\mathbf{v} - L\mathbf{x})$
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The multivelel direct method

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$$\mathrm{W} - \mathrm{cycle}$$

The multivelel direct method

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W – cycle

K – cycle

Conclusion

Thank you for your attention

Numerical example

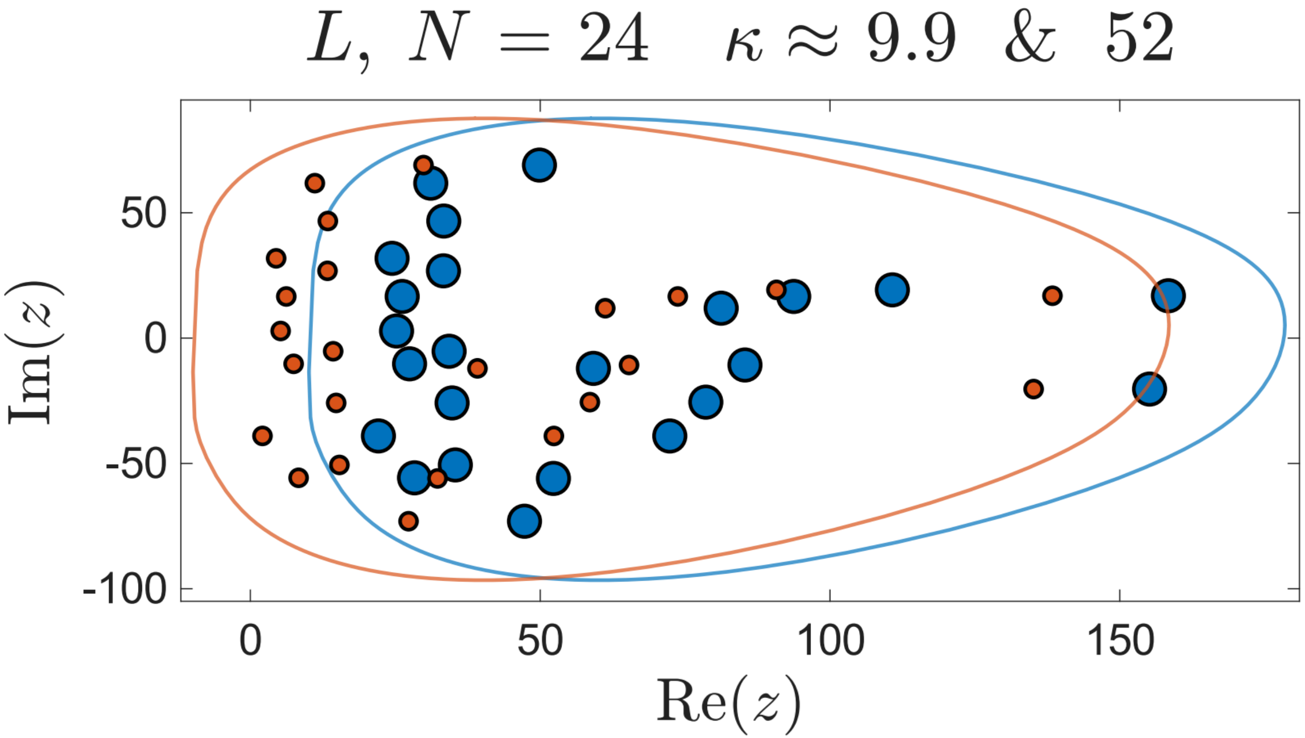
Numerical example

https://pablo.world/CCIMM26/#/5

Numerical example

$$L = H + \gamma K$$

Numerical example



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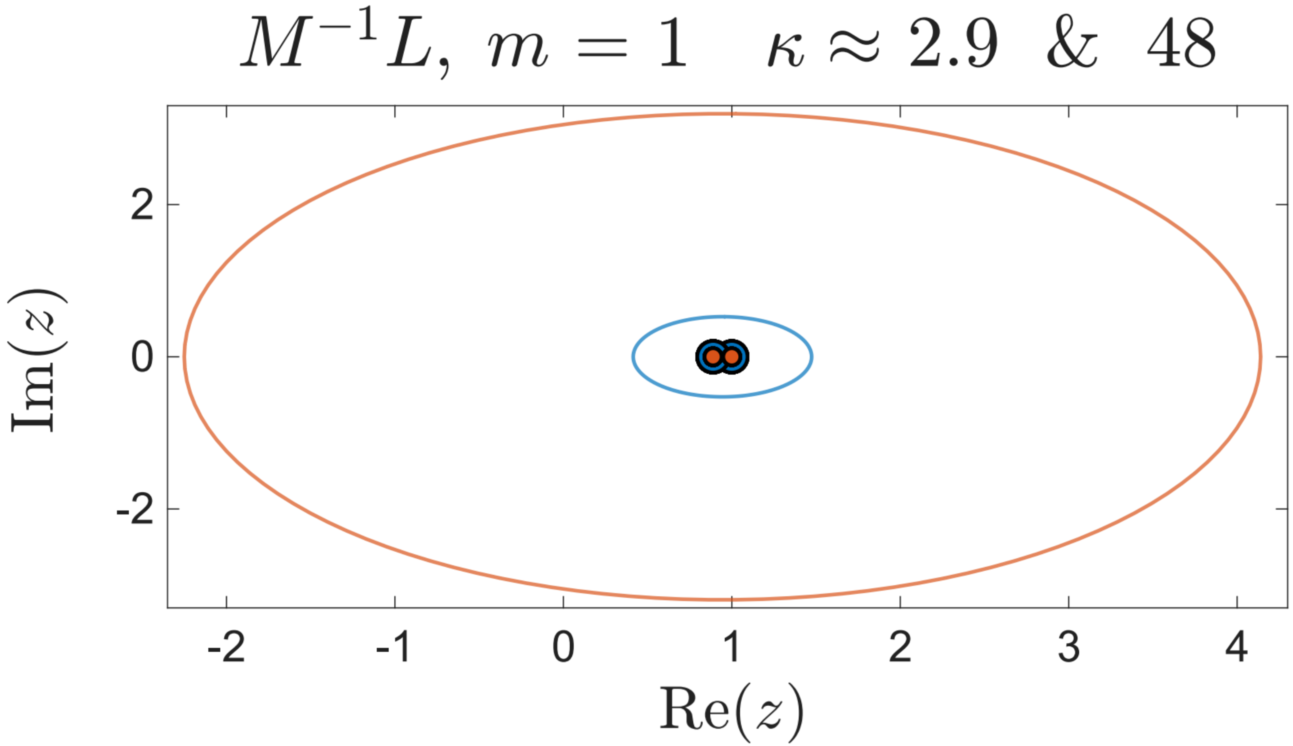

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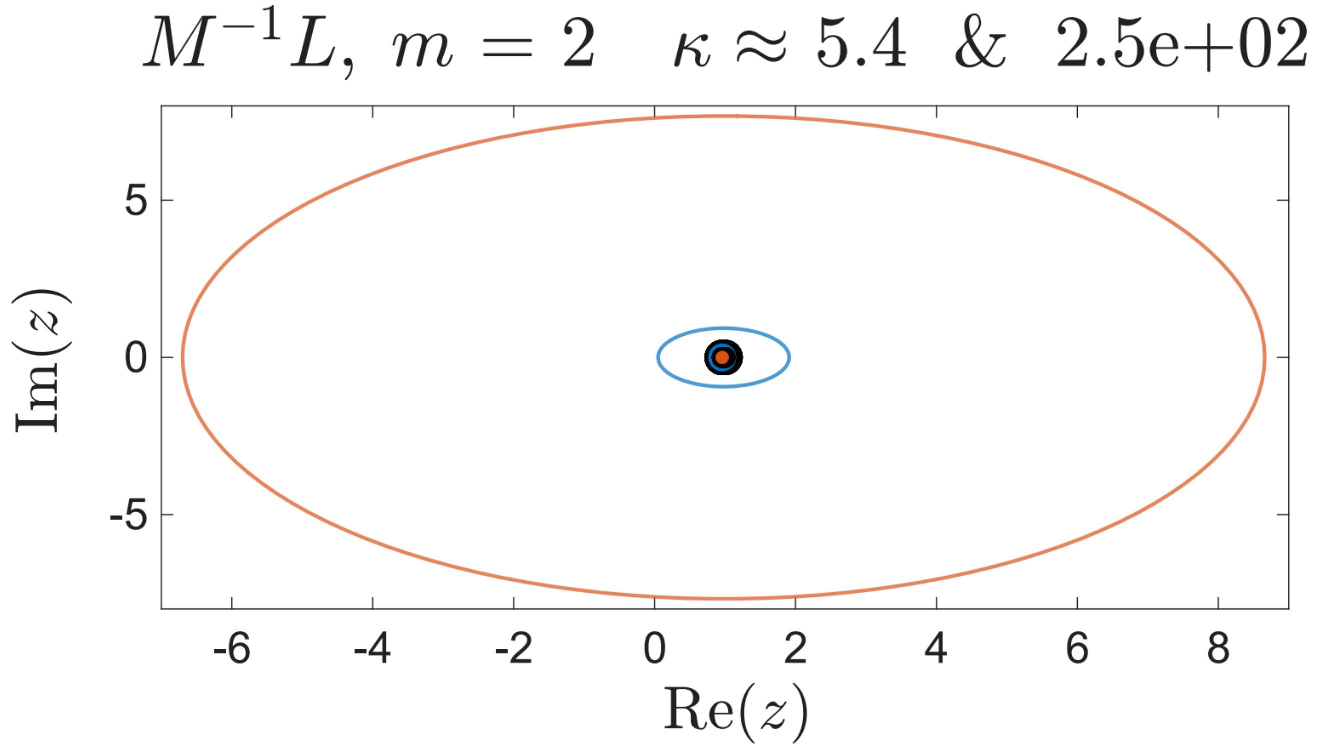

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