Multiprecision computations with Schwarz methods

Michal Outrata

with Daniel B. Szyld (Temple)

Outline

- Model problem and set-up
- Schwarz methods and convergence
- Multiprecision
- Results

Model problem

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$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial\Omega
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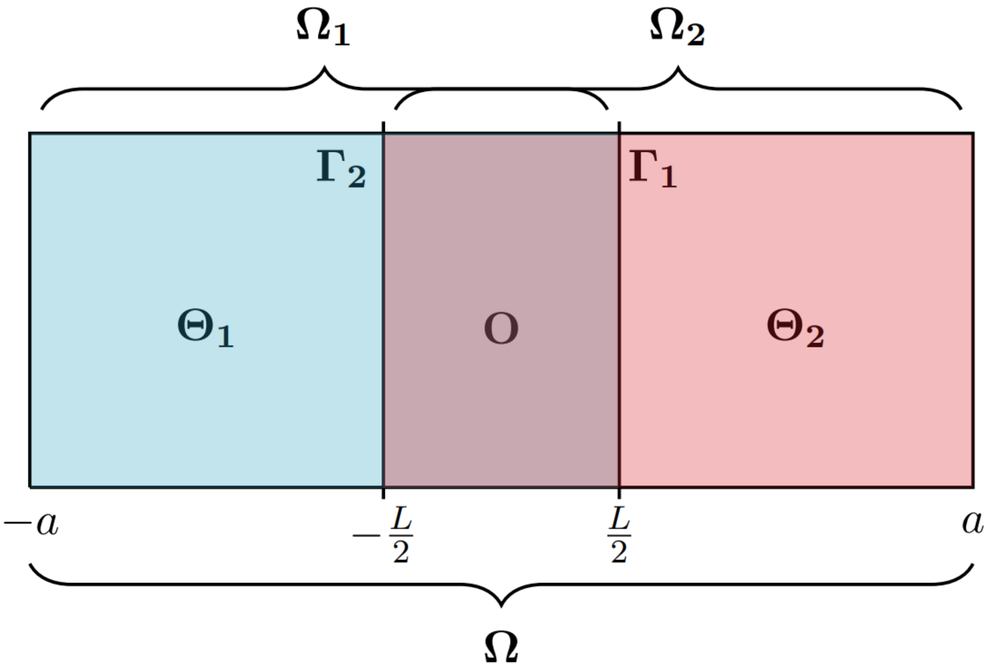
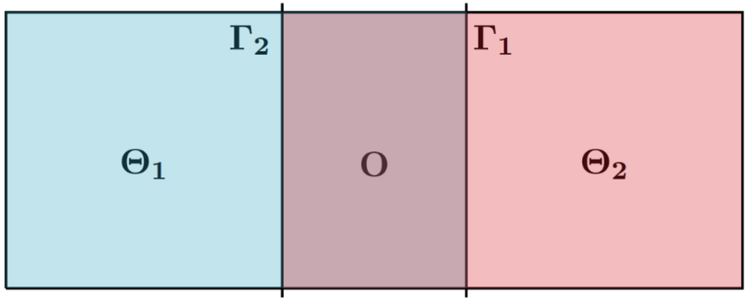


Schwarz methods

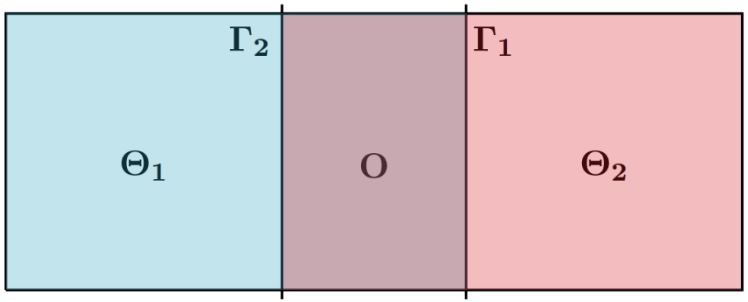
Schwarz methods

$u^{(0)}$



Schwarz methods

$$u^{(0)} \rightarrow \boxed{(u^{(0)})_{\Gamma_1}} \rightarrow \boxed{u_1^{(1)}} \rightarrow \boxed{(u_1^{(1)})_{\Gamma_2}} \rightarrow \boxed{u_2^{(1)}}$$



Why does it work?

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- Fourier analysis of the contraction
- Variational bounds of the condition number
- Maximum principle

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Discretized problem

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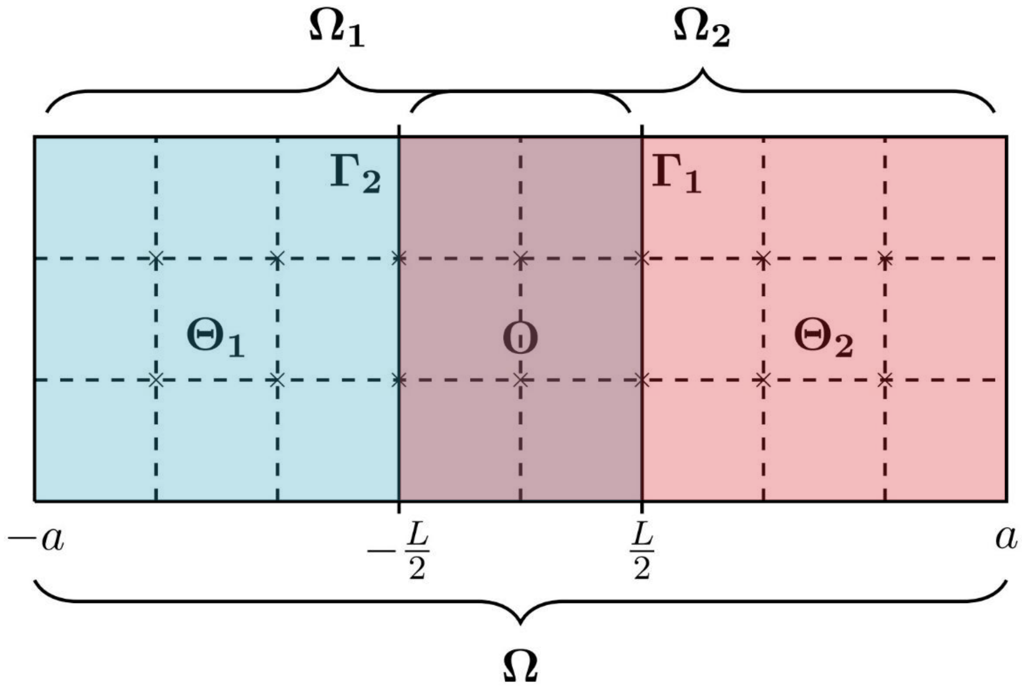
$$A\mathbf{u} = \mathbf{b}$$



Schwarz methods

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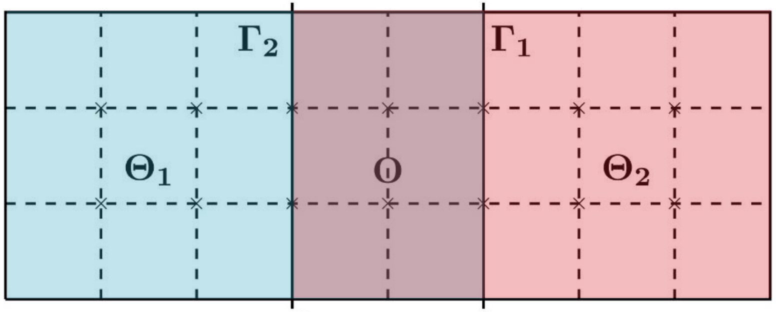

Why does it work?

Why does it work?

- Diagonalization
- Bounds of the condition number & CG
- M-matrices & matrix splitting

M-matrices

M-matrices

Definition. *Let* $A \in \mathbb{R}^{N\times N}$ *be non-singular. We say* A *is an* M*-matrix, provided that*

$$A_{ij} \le 0, \quad \forall\, i \neq j \qquad \& \qquad \left(A^{-1}\right)_{ij} \ge 0, \quad \forall\, i, j.$$

M-matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{g} \end{bmatrix}$$

M-matrices

Theorem. *Let* $A_{11} \in \mathbb{R}^{n_1 \times n_1}, A_{12} \in \mathbb{R}^{n_1 \times n_2}$ *be such that*

- A_{11} *is non-singular and irreducible,*
- $A_{12} \leq 0$ *and each column has at least one non-zero entry,*
- *the row sums of* $[A_{11}\ A_{12}]$ *are all zero.*

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is an M*-matrix, then* $\mathbf{x}$ *satisfies the discrete maximum principle, i.e.,*

$$\min_{k=1,\ldots,n_2} g_k \leq x_i \leq \max_{k=1,\ldots,n_2} g_k \quad \text{for all } i = 1, \ldots, n_1 + n_2.$$

Multiprecision algebraic Schwarz

Multiprecision algebraic Schwarz

$$\frac{1}{h^2}\begin{bmatrix} D & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & D \end{bmatrix} \mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

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Carry out the expensive part in low-precision

Multiprecision algebraic Schwarz

Existing approaches & mixed-precision

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Multiprecision algebraic Schwarz

$\boxed{\text{How}}$ to round & $\boxed{\text{how much?}}$

Multiprecision algebraic Schwarz

How?

Multiprecision algebraic Schwarz

How?

	Signif.	Exp.	u	x_{min}	x_{max}
q52	5	2	1.25×10^{-1}	6.10×10^{-5}	5.73×10^{4}
q43	4	3	6.25×10^{-2}	1.56×10^{-2}	2.40×10^{2}
bfloat16	8	8	3.91×10^{-3}	1.18×10^{-38}	3.39×10^{38}
fp16	11	5	4.88×10^{-4}	6.10×10^{-5}	6.55×10^{4}
fp32	24	8	5.96×10^{-8}	1.18×10^{-38}	3.40×10^{38}
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Step 1 – rescale to range

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$$A\mathbf{x} = \mathbf{b} \quad \longrightarrow \quad \mathcal{A}\mathbf{u} = \mathbf{f}$$

Multiprecision algebraic Schwarz

How?

Step 2 – preserve the M-matrix property

Multiprecision algebraic Schwarz

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Step 2 – preserve the M-matrix property

$$\mathcal{A}_{ij} \leq 0, \quad \forall\, i \neq j \qquad \& \qquad \left(\mathcal{A}^{-1}\right)_{ij} \geq 0, \quad \forall\, i, j$$

Multiprecision algebraic Schwarz

How?

Step 2 – preserve the M-matrix property

$$\mathcal{A}_{ij} \leq 0, \quad \forall\, i \neq j \qquad \& \qquad \left(\mathcal{A}^{-1}\right)_{ij} \geq 0, \quad \forall\, i, j$$

$$\mathcal{A}\mathbf{u} = \mathbf{f} \qquad \longrightarrow \qquad \tilde{\mathcal{A}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}$$

Multiprecision algebraic Schwarz

How much?

Multiprecision algebraic Schwarz

How much?

Theorem. *Let* $\mathtt{F}_i := \tilde{\mathcal{A}} - \mathcal{A}$ *and assume that*

$$\|\mathcal{A}_i^{-1}\mathtt{F}_i\| < 1 \quad \text{and} \quad \mathcal{A}_i^{-1} \geq \mathcal{A}_i^{-1}\mathtt{F}_i\mathcal{A}_i^{-1}.$$

Then, the multiprecision versions of the classical Schwarz methods converge.

Multiprecision algebraic Schwarz

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sharp? improvement? diminishing returns?

Numerical results

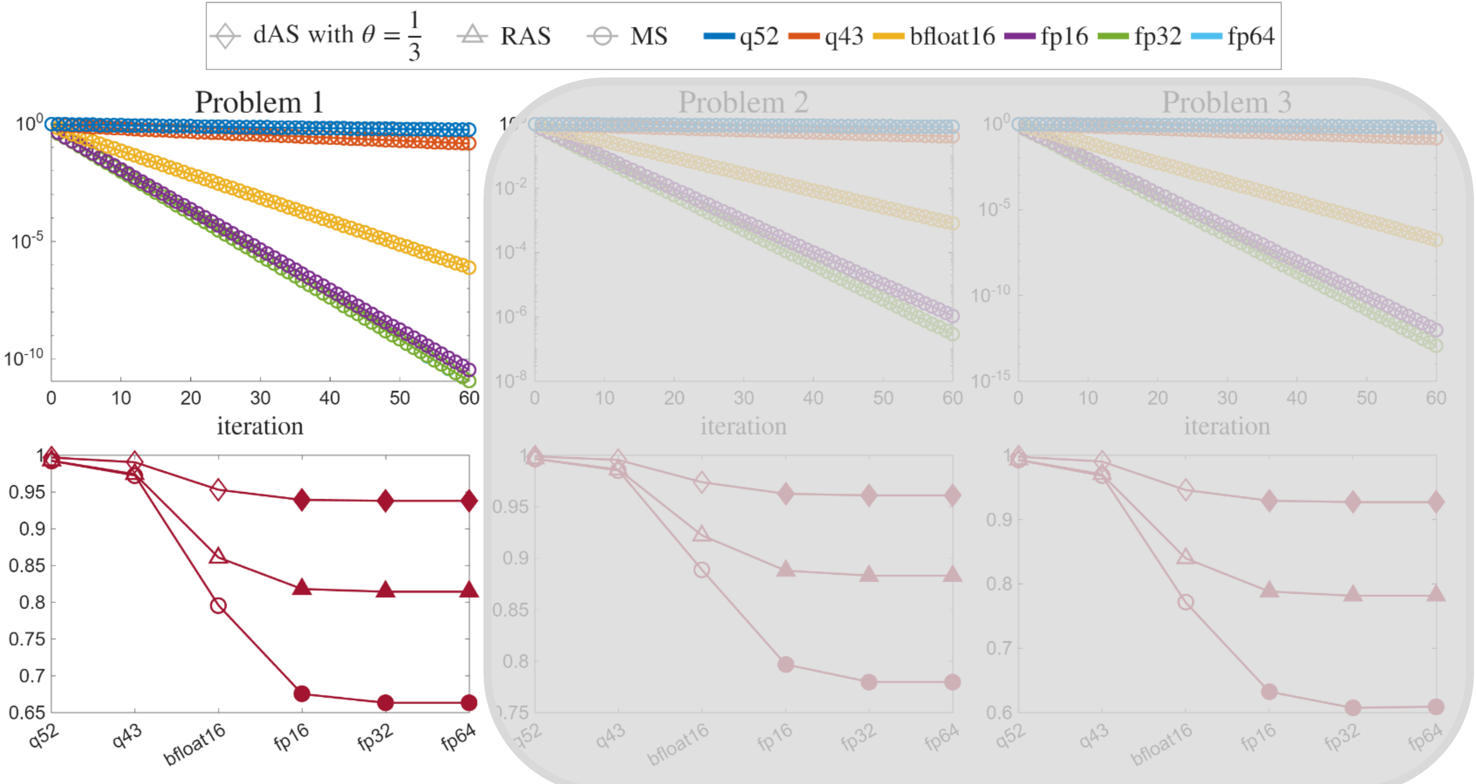
Numerical results

$$\eta(\mathbf{x})u - \operatorname{div}\left(\alpha(\mathbf{x})\nabla u\right) + \mathbf{b}(\mathbf{x}) \cdot \nabla u$$

$$\eta(\mathbf{x}) := x_1^2 \cos(x_1 + x_2)^2, \;\; \alpha(\mathbf{x}) := 20(x_1 + x_2)^2 e^{x_1 - x_2}$$

$$b_1(\mathbf{x}) := x_2 - 0.5, \;\; b_2(\mathbf{x}) := x_1 - 0.5.$$

Numerical results



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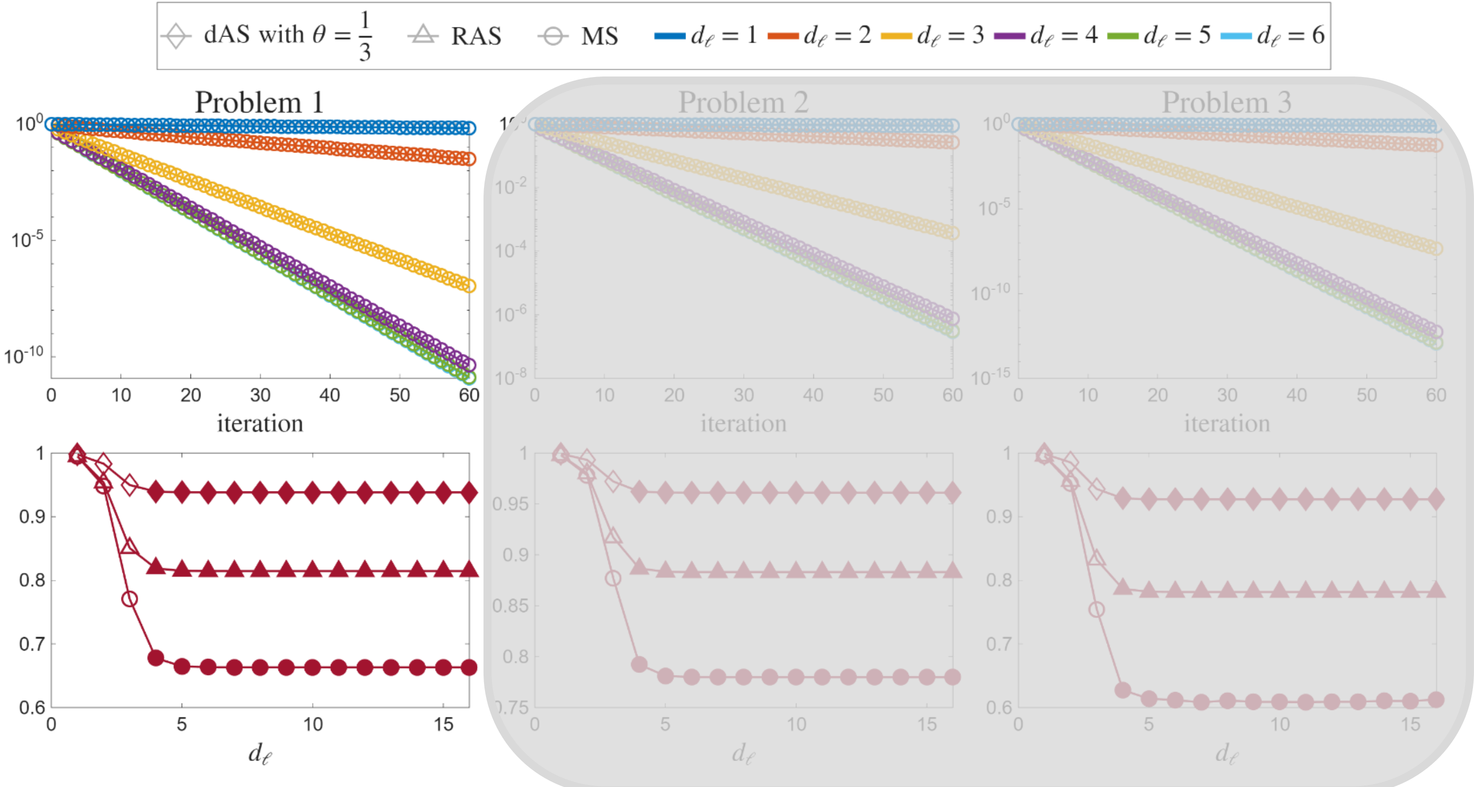

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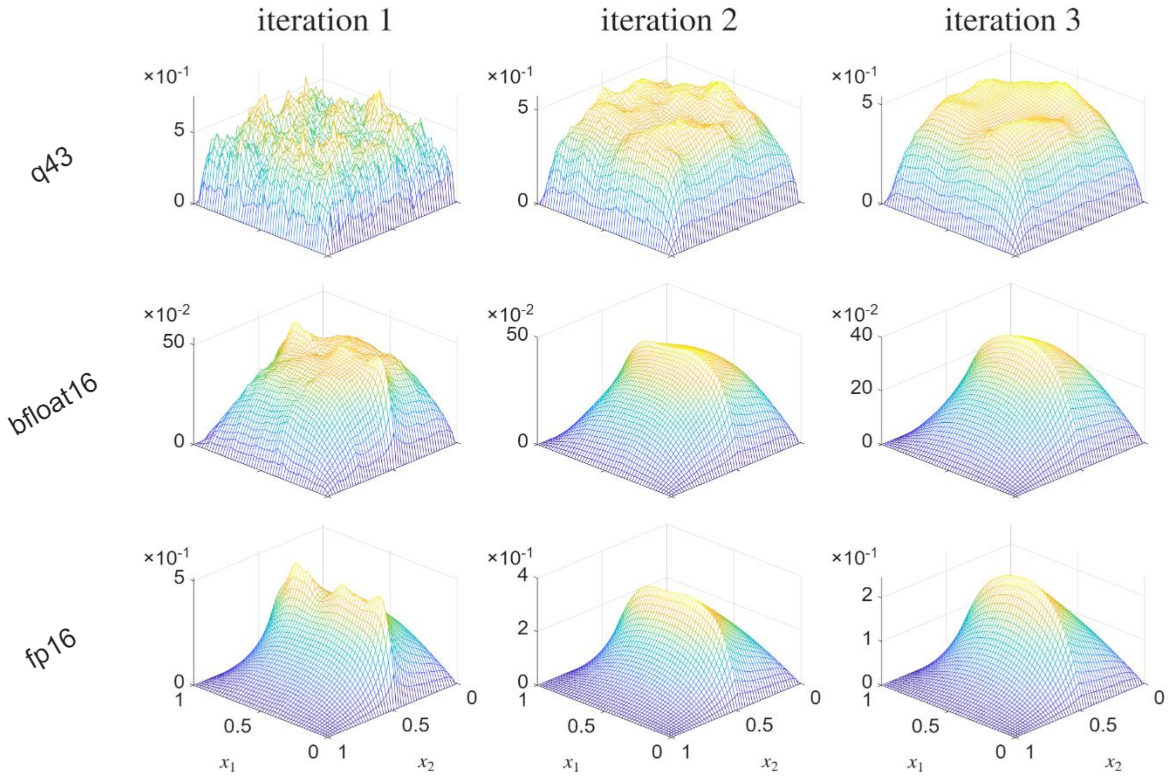

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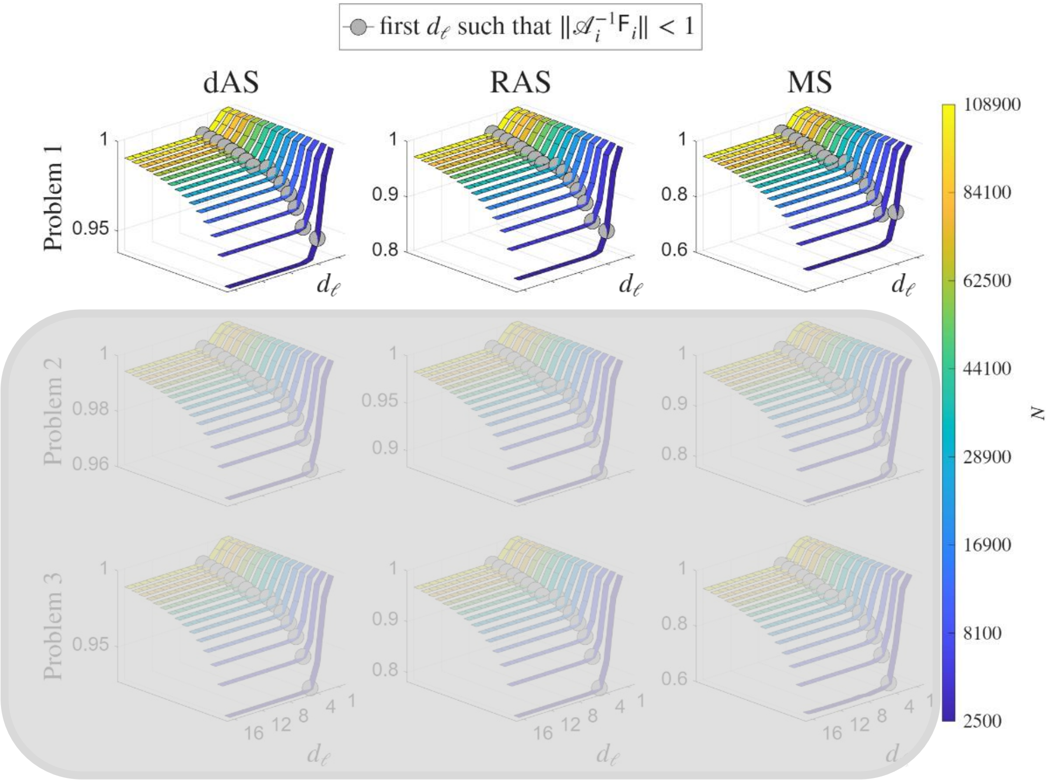

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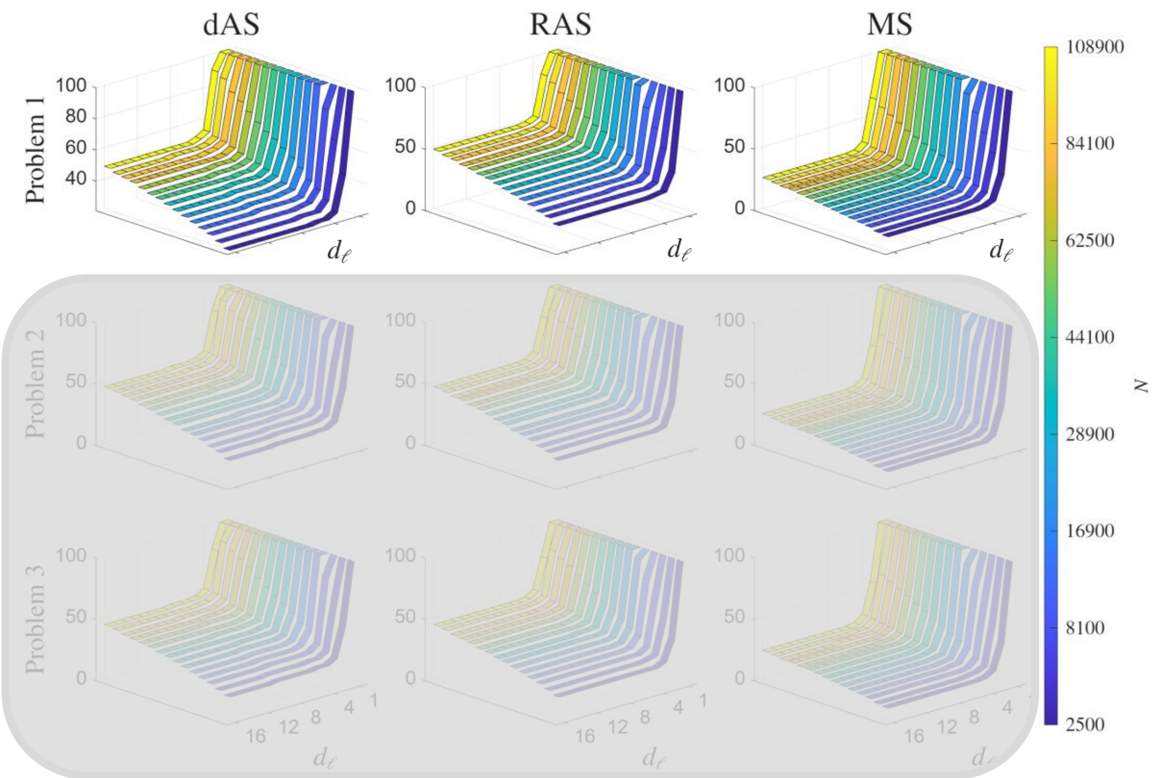


Conclusion

Thank you for your attention

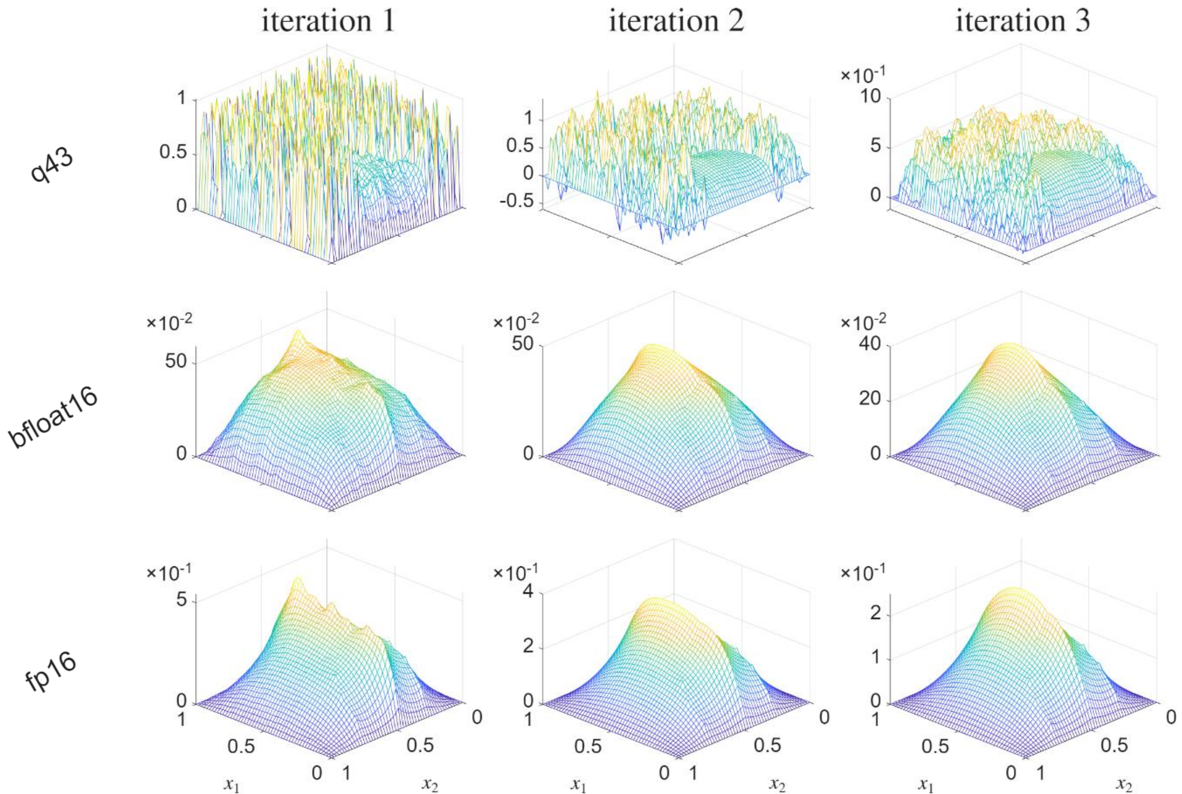
Numerical results

$$\eta(\mathbf{x})u - \operatorname{div}\left(\alpha(\mathbf{x})\nabla u\right) + \mathbf{b}(\mathbf{x}) \cdot \nabla u$$

$$\eta \equiv 0, \;\; \alpha(\mathbf{x}) = \begin{cases} 10^6 & \text{if } \|\mathbf{x} - [0.5\ 0.1]^T\| < 0.25, \\ 1 & \text{otherwise,} \end{cases}$$

$$b_1(\mathbf{x}) = b_2(\mathbf{x}) = 0$$

Numerical results



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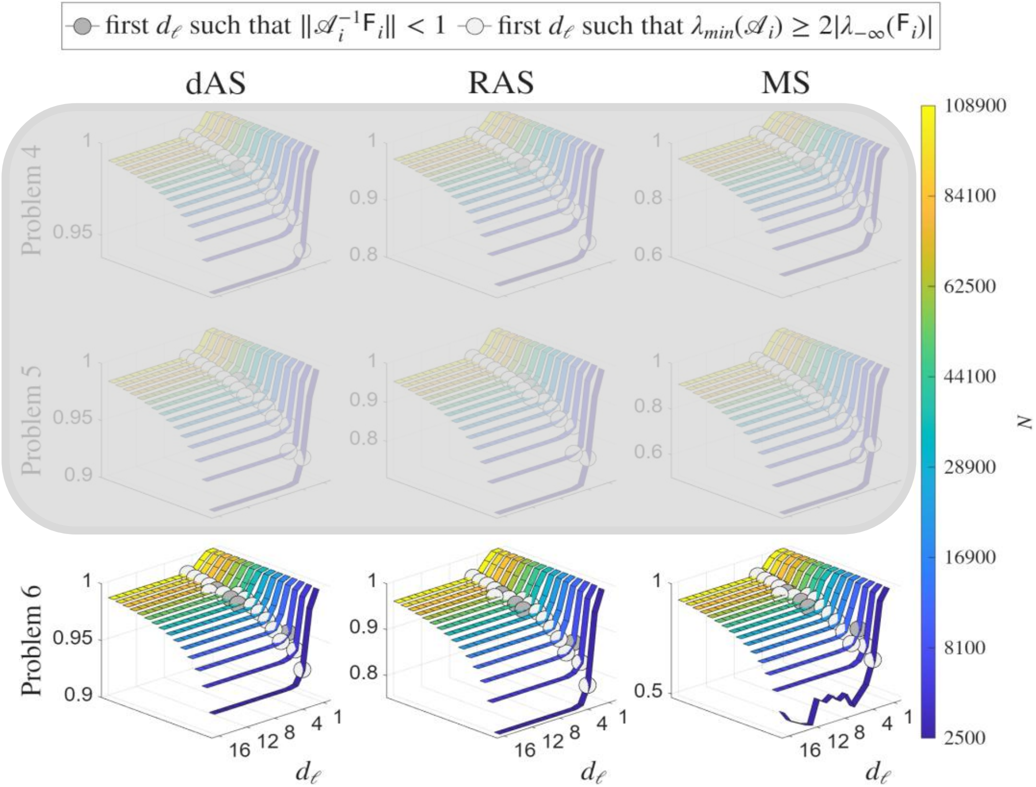

Numerical results