Domain truncation, absorbing BCs, Schur complement and Padé approximants

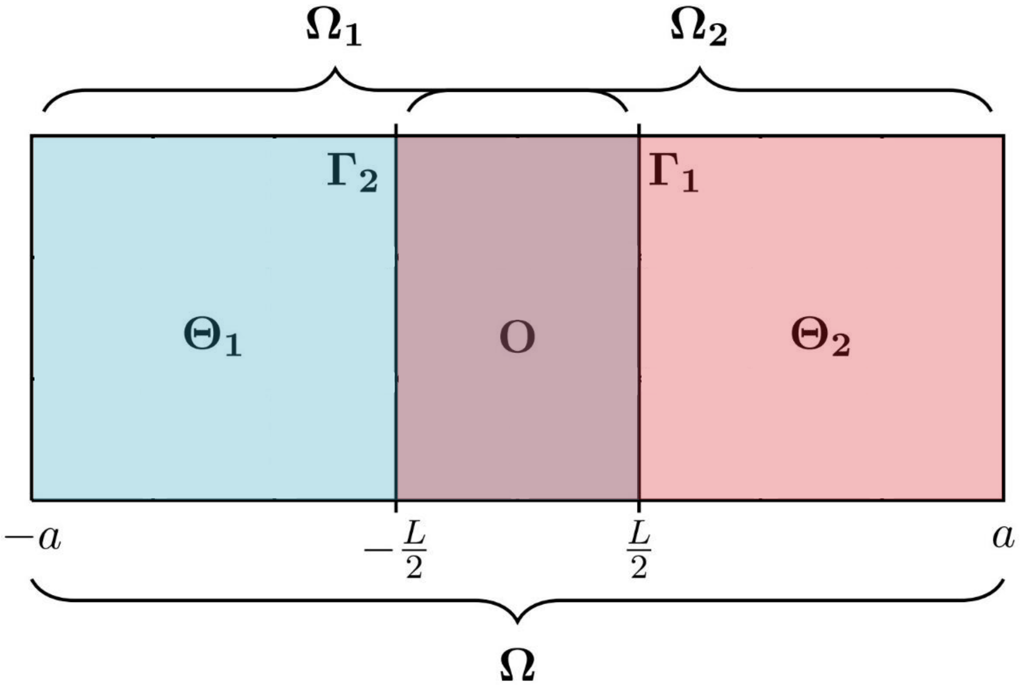
Michal Outrata

with Martin J. Gander (UNIGE)

Outline

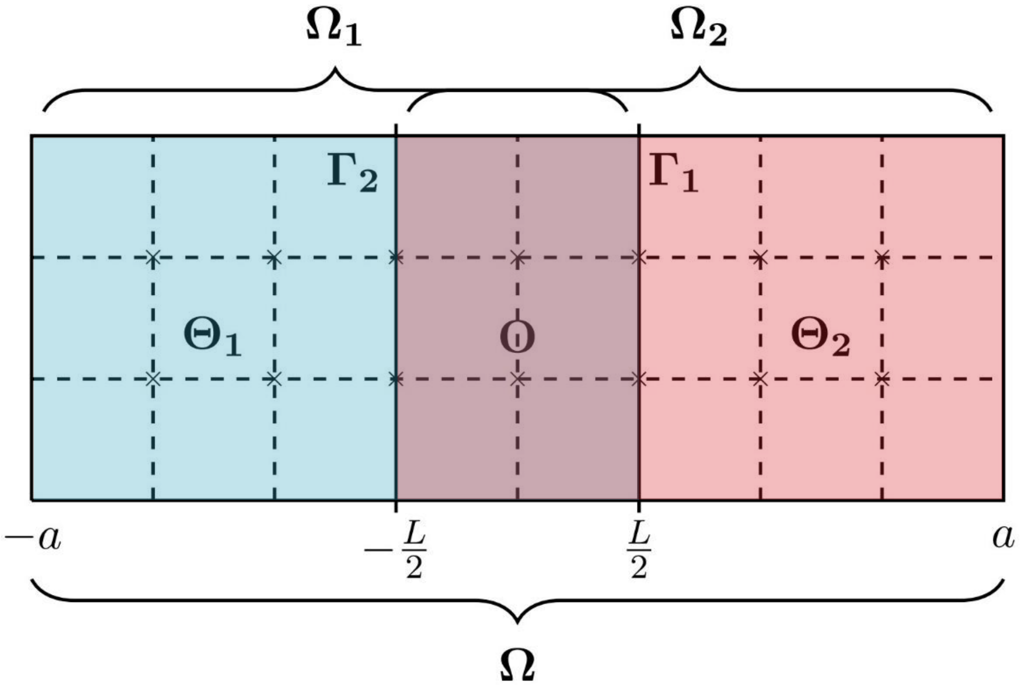
- Model problem and set-up
- Schwarz methods and ABC
- ABC analysis

Model problem



$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial\Omega
\end{aligned}
$$

Model problem

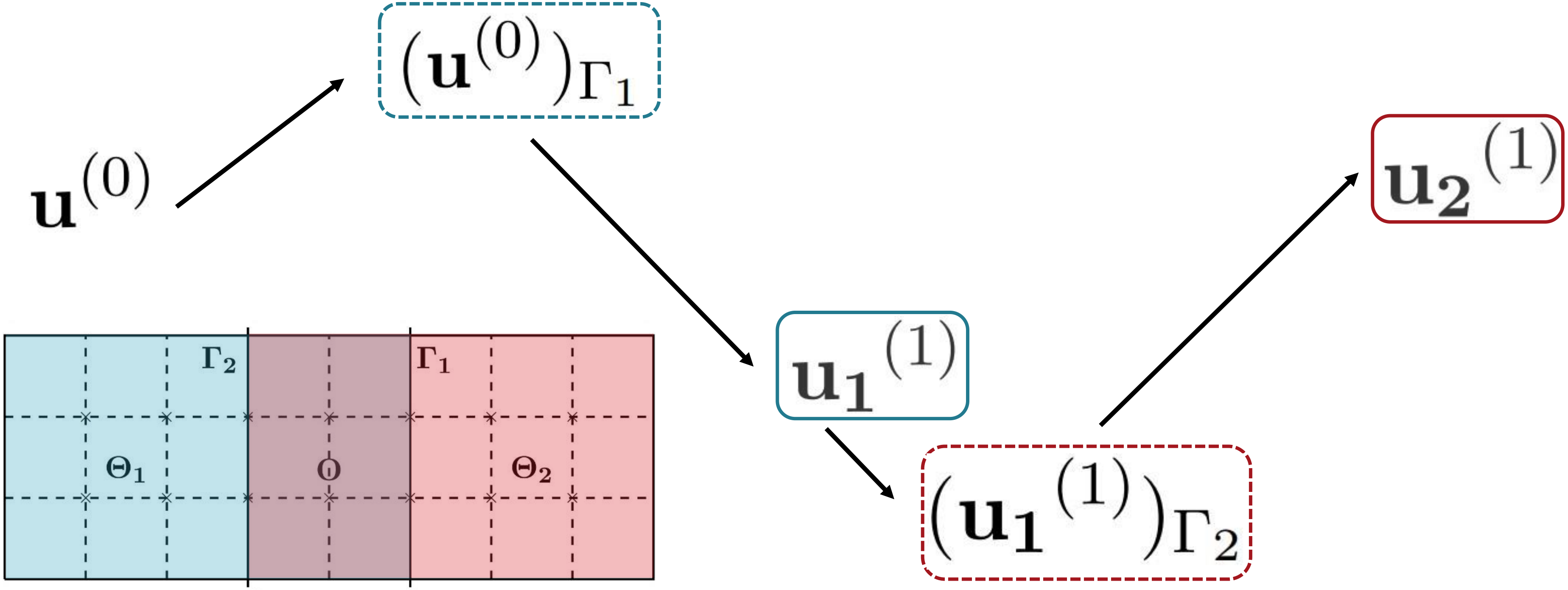


$$L\mathbf{u} = \mathbf{f}$$

blackboard

Schwarz methods

Schwarz methods



Schwarz methods

$$\frac{1}{h^2}\begin{bmatrix} D & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & D \end{bmatrix} \mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & D \end{bmatrix} \mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

Schwarz methods

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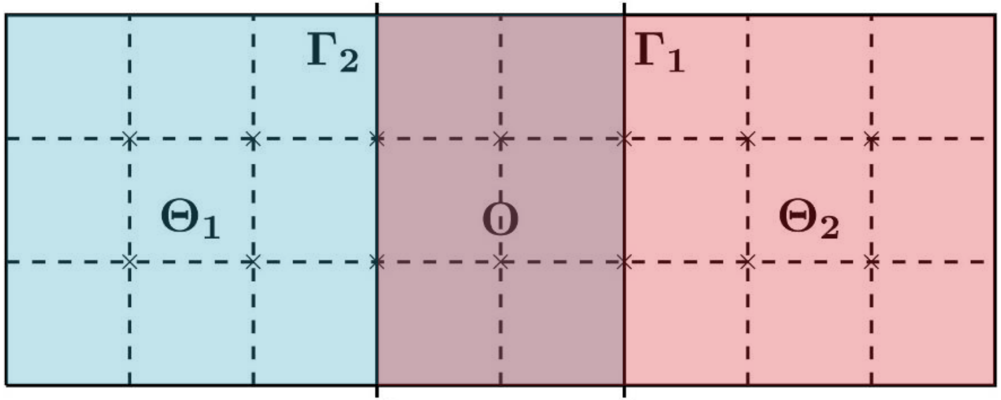
$$\rho = \frac{\sinh\left(\frac{\pi}{b}(a - \frac{L}{2})\right)}{\sinh\left(\frac{\pi}{b}(a + \frac{L}{2})\right)}$$

Optimal Schwarz methods

Optimal Schwarz methods



Optimal Schwarz methods

$$\frac{1}{h^2}\begin{bmatrix} D & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D & -I \\ & & -I & D - S^\star \end{bmatrix}\mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

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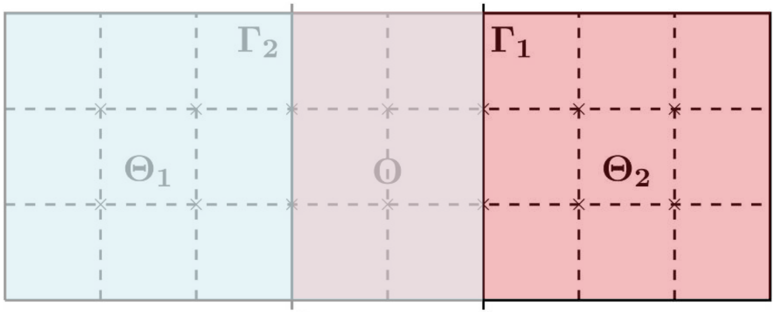
$$S^\star := E_{\Gamma_1}^T L_{\Theta_2}^{-1} E_{\Gamma_1}$$

Optimal Schwarz methods

$$S^{\star} := E_{\Gamma_1}^{T} L_{\Theta_2}^{-1} E_{\Gamma_1}$$



Optimized Schwarz methods

Optimized Schwarz methods

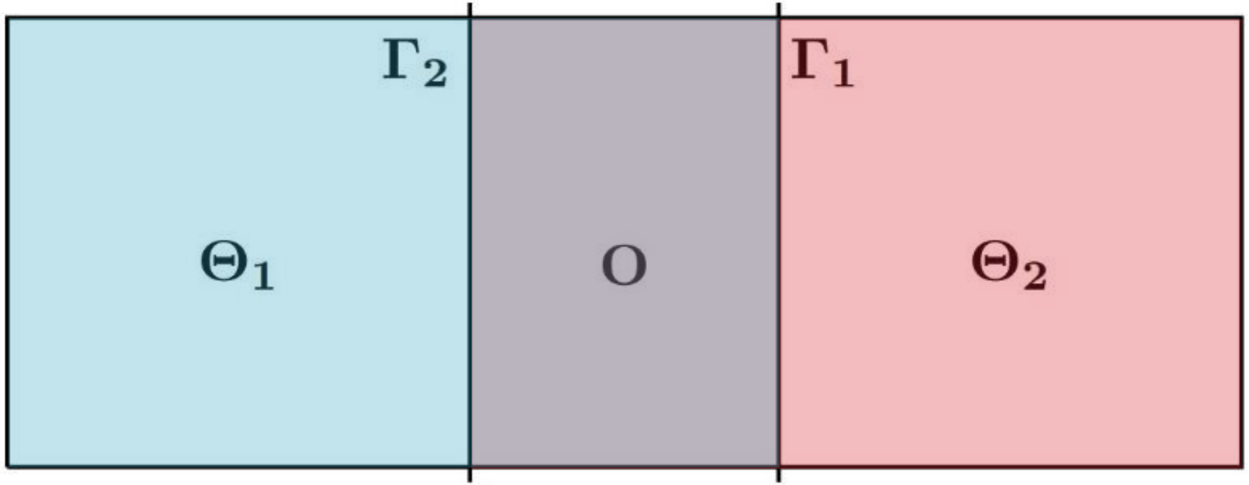
$$\frac{1}{h^2}\begin{bmatrix} D & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D & -I \\ & & -I & D-S \end{bmatrix}\mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D-S & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D & -I \\ & & -I & D \end{bmatrix}\mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

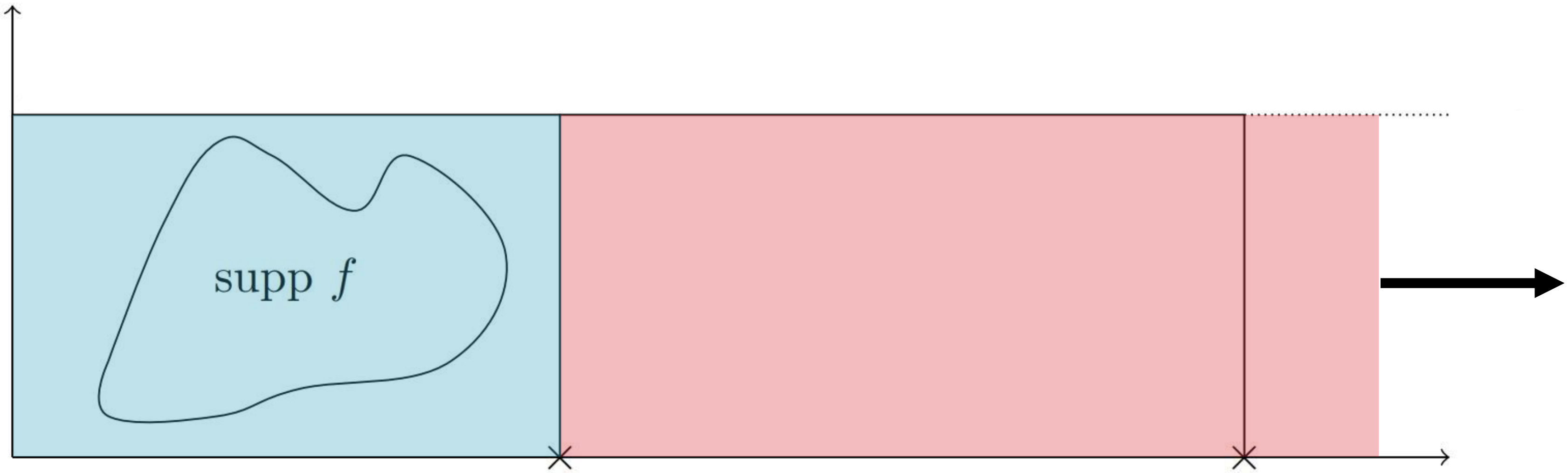
$$S^\star \to S$$

Schwarz methods & ABC

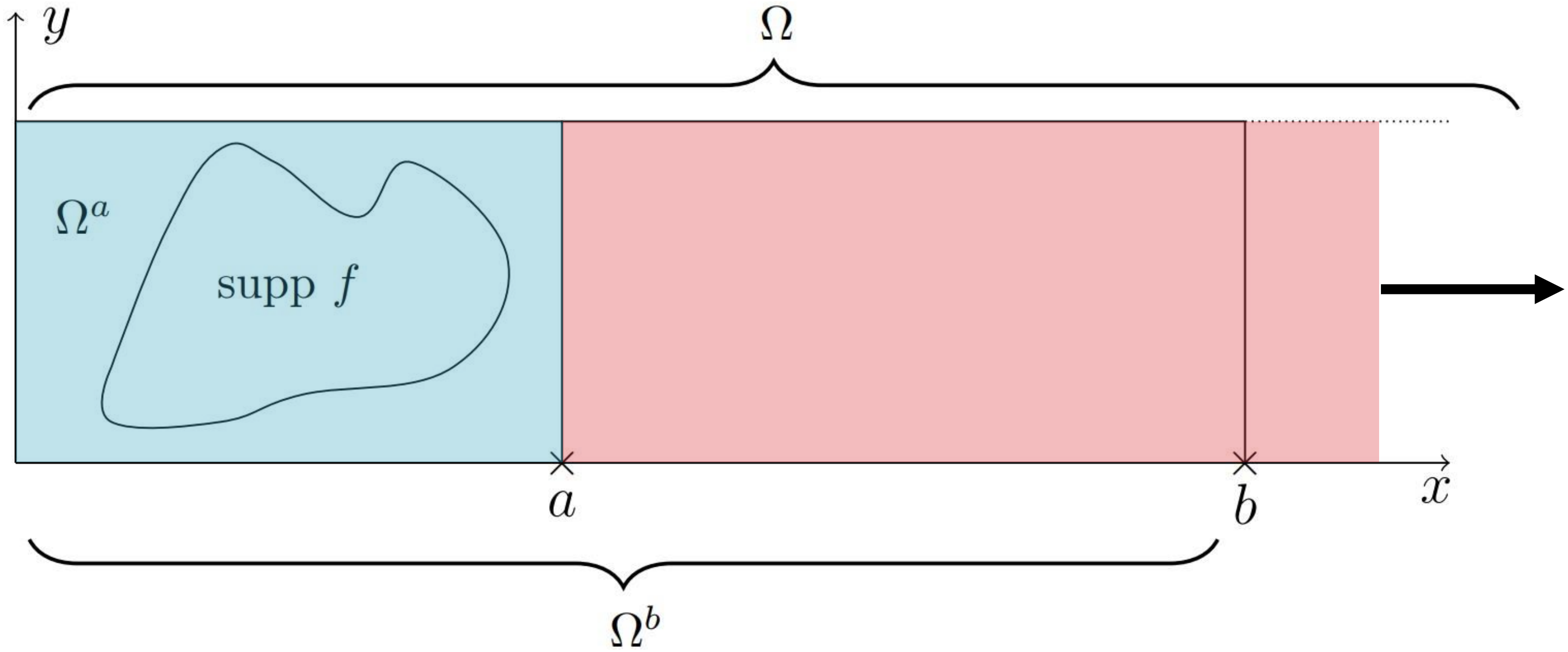
Schwarz methods & ABC



Schwarz methods & ABC



Schwarz methods & ABC



Discrete ABCs

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$$L^a \mathbf{u}^a = \mathbf{f}^a \qquad L^b \mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

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$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix} \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^b-1} & -I \\ & & -I & D_{N^b} \end{pmatrix}$$

where $D_i = D$ (blackboard)

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where $D_i = D$ (blackboard)

Discrete ABCs

$$L^a \mathbf{u}^a = \mathbf{f}^a \qquad L^b \mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix} \quad \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & \boxed{T^b_{N^a}} \end{pmatrix} \quad \begin{pmatrix} h^2 L^b & & \\ & -I & \\ -I & D_{N^b+1} & \ddots \\ & \ddots & \ddots \end{pmatrix}$$

$T^b_{N^a}$ (blackboard)

Discrete ABCs

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$T^b_{N^a}, T^\infty_{N^a}$ (blackboard)

Discrete ABCs

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How to describe the effect of moving b ?

Fourier analysis & CF

Fourier analysis & CF

$T^b_{N^a}$:

Fourier analysis & CF

$$\hat{T}_i^b = Q\frac{D}{h^2}Q^T - Q\frac{(T_{i+1}^b)^{-1}}{h^4}Q^T = \frac{\Lambda}{h^2} - \frac{(\hat{T}_{i+1}^b)^{-1}}{h^4}$$

$T_{N^a}^b$:

Fourier analysis & CF

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$T_{N^a}^b$:

$$\hat{t}_i^b(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2\hat{t}_{i+1}^b(\lambda)}\right)$$

Fourier analysis & CF

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Fourier analysis & CF

$T^b_{N^a}$:

$$\hat{t}^b_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \cfrac{1}{\lambda - \cfrac{\ddots}{\lambda - \frac{1}{\lambda}}}\right)$$

$N^b - N^a$ levels; blackboard

Fourier analysis & CF

$T^{\infty}_{N^a}$:

Fourier analysis & CF

$$\hat{T}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{T}^{b}_{N^a}(\lambda)$$

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Fourier analysis & CF

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Fourier analysis & CF

$T^{\infty}_{N^a}$:

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \cfrac{1}{\lambda - \cfrac{\ddots}{\lambda - \cfrac{1}{\ddots}}}\right)$$

blackboard

Fourier analysis & CF

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$T^{\infty}_{N^a}$:

Fourier analysis & CF

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2\hat{t}^{\infty}_{N^a}(\lambda)}\right)$$

$T^{\infty}_{N^a}$:

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2h^2}$$

blackboard

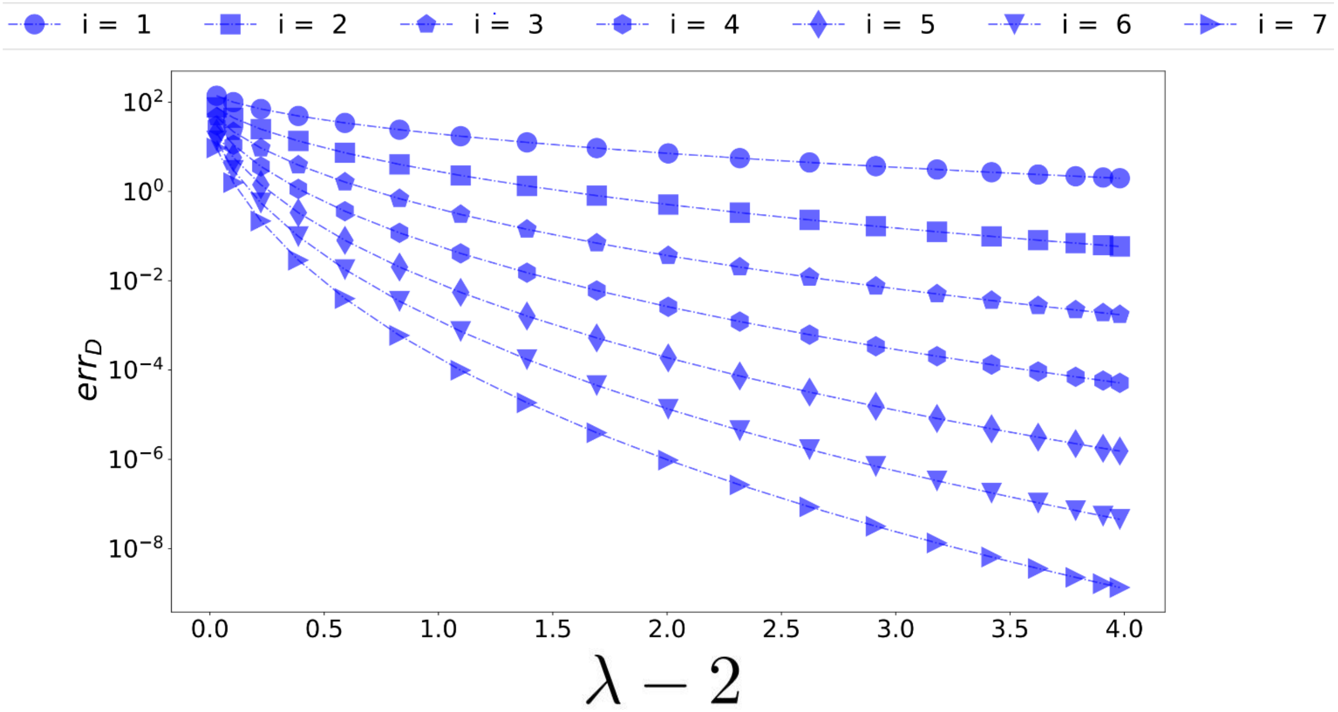
Understanding ABC

Understanding ABC

$T^{b}_{N^a}$ vs $T^{\infty}_{N^a}$

How to describe the effect of moving b ?

Understanding ABC



Continued fractions and *Padé*

Continued fractions and *Padé*

$$f \approx \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}$$

Continued fractions and Padé

Continued fractions and Padé

Theorem. *For any* $\alpha \in (-1, +\infty)$ *we have*

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

Moreover, for any n *the* $[n+1, n]$*-Padé approximant of* $\sqrt{1+\alpha}$ *expanded about* $\alpha = 0$ *is the* $(2n+1)$*-st truncation of the continued fraction above.*

Continued fractions and Padé

Combine

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

Continued fractions and Padé

Combine

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

and

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2h^2}$$

Continued fractions and Padé

To combine

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

and

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... need some dress-up ...

Continued fractions and Padé

Having $\lambda = 2 + z$ we get

Continued fractions and Padé

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$$\hat{t}^{\infty}_{N^a}(z) = \frac{1}{h^2}\left(1 + \frac{z}{2} + \frac{z}{2}\sqrt{1 + \frac{4}{z}}\right)$$

Continued fractions and Padé

Having $\lambda = 2 + z$ we get

$$\hat{t}^{\infty}_{N^a}(z) = \frac{1}{h^2}\left(1 + \frac{z}{2} + \frac{z}{2}\sqrt{1 + \frac{4}{z}}\right) = \frac{1}{h^2}\left(2 + z - \cfrac{1}{2 + z - \cfrac{1}{2+z-\cfrac{1}{2+z-\frac{1}{\cdots}}}}\right)$$

Continued fractions and Padé

Having $\lambda = 2 + z$ we get

$$\hat{t}^{\infty}_{N^a}(z) = \frac{1}{h^2}\left(1 + \frac{z}{2} + \frac{z}{2}\sqrt{1+\frac{4}{z}}\right) = \frac{1}{h^2}\left(2 + z - \frac{1}{2 + z - \frac{1}{2+z-\frac{1}{2+z-\frac{1}{\cdots}}}}\right)$$

$$\hat{t}^{b}_{N^a}(z) = \frac{1}{h^2}\left(2 + z - \frac{1}{2 + z - \frac{1}{\ddots_{\,2+z-\frac{\ddots}{2+z-\frac{1}{2+z}}}}}\right)$$

Continued fractions and Padé

Combining

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

Continued fractions and Padé

Combining

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and

$$\hat{t}^{\infty}_{N^a}(z) = \frac{1}{h^2}\left(1 + \frac{z}{2} + \frac{z}{2}\sqrt{1+\frac{4}{z}}\right)$$

... after some tedious fraction manipulations ...

Understanding ABC

Having $\lambda = 2 + z$ we get

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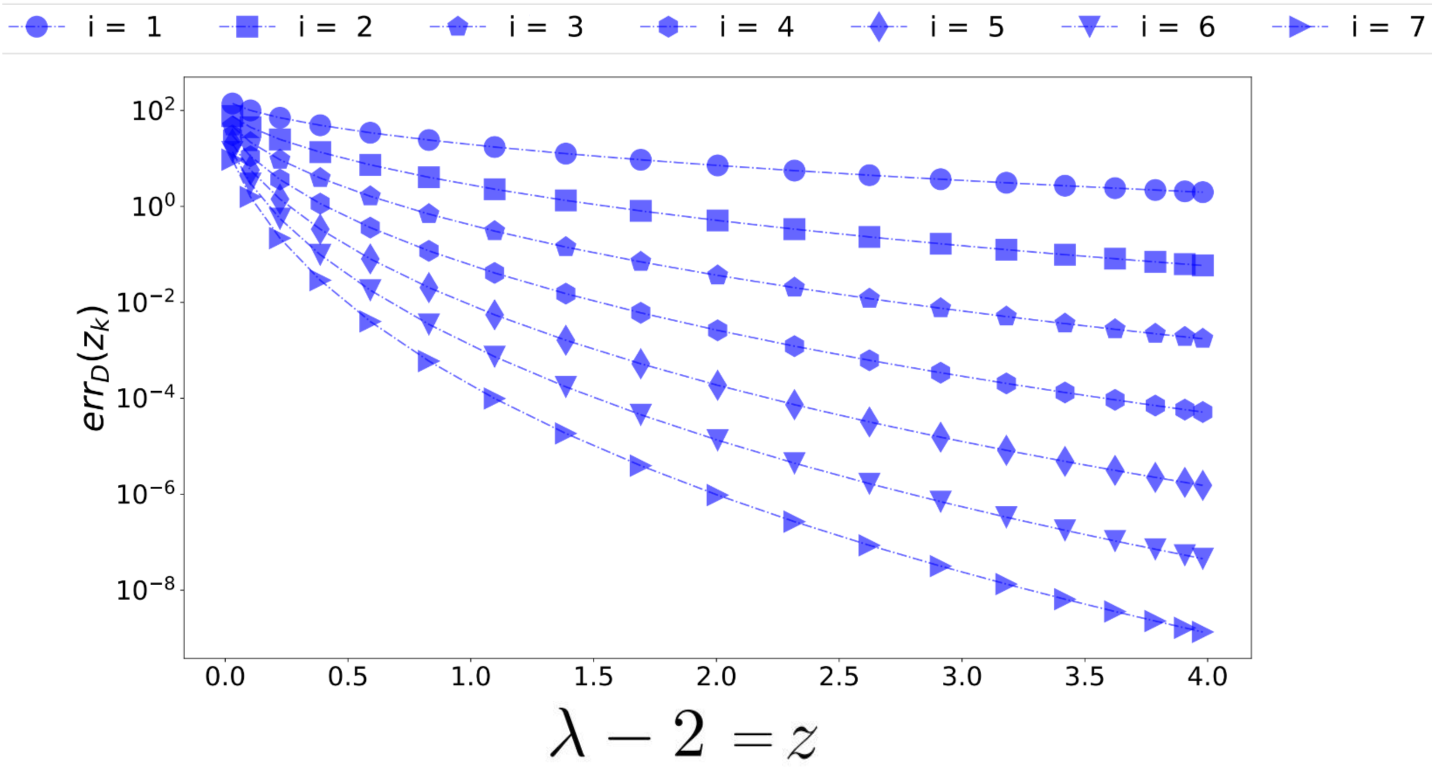
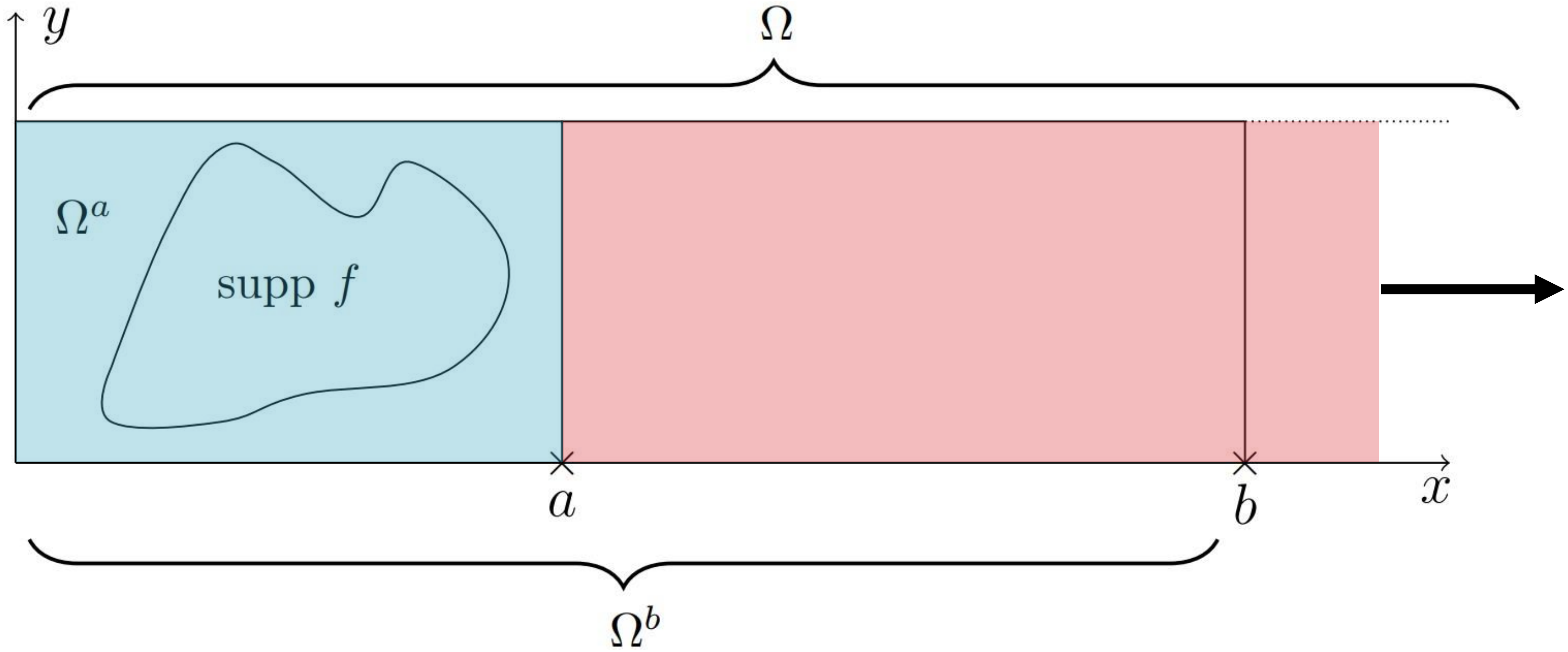
Theorem. *The function* $\hat{t}^{b}_{N^a}(z)$ *is the* $[\,i\,,\,i\,]$*-Padé approximation about the expansion point* $z = +\infty$ *of* $\hat{t}^{\infty}_{N^a}(z)$, *where* $i = N^b - N^a$.

Understanding ABC

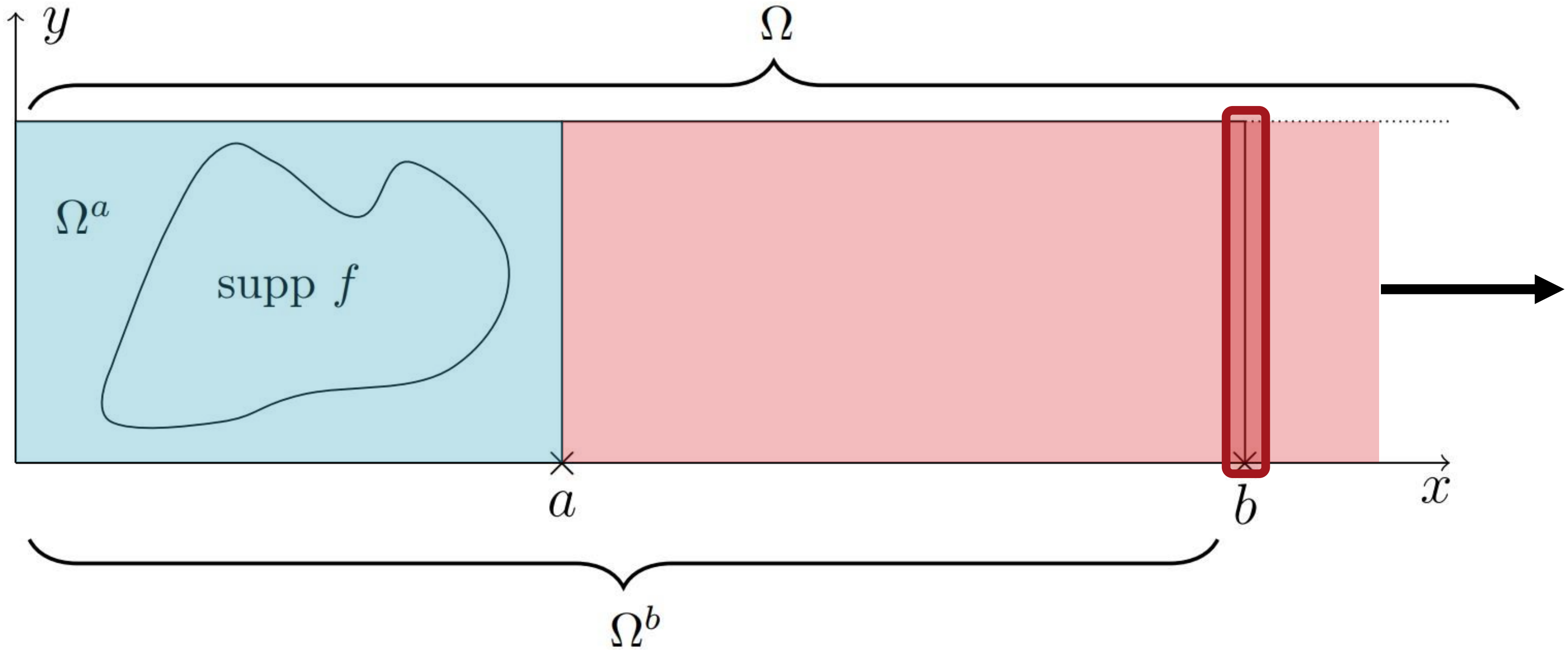


Improving ABC

Improving ABC – Robin



Improving ABC – Robin



Improving ABC – Robin

$$L^b \mathbf{u}^b = \mathbf{f}^b$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^b-1} & -I \\ & & -I & \boxed{D_{N^b}} \end{pmatrix}$$

Improving ABC – Robin

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Improving ABC – Robin

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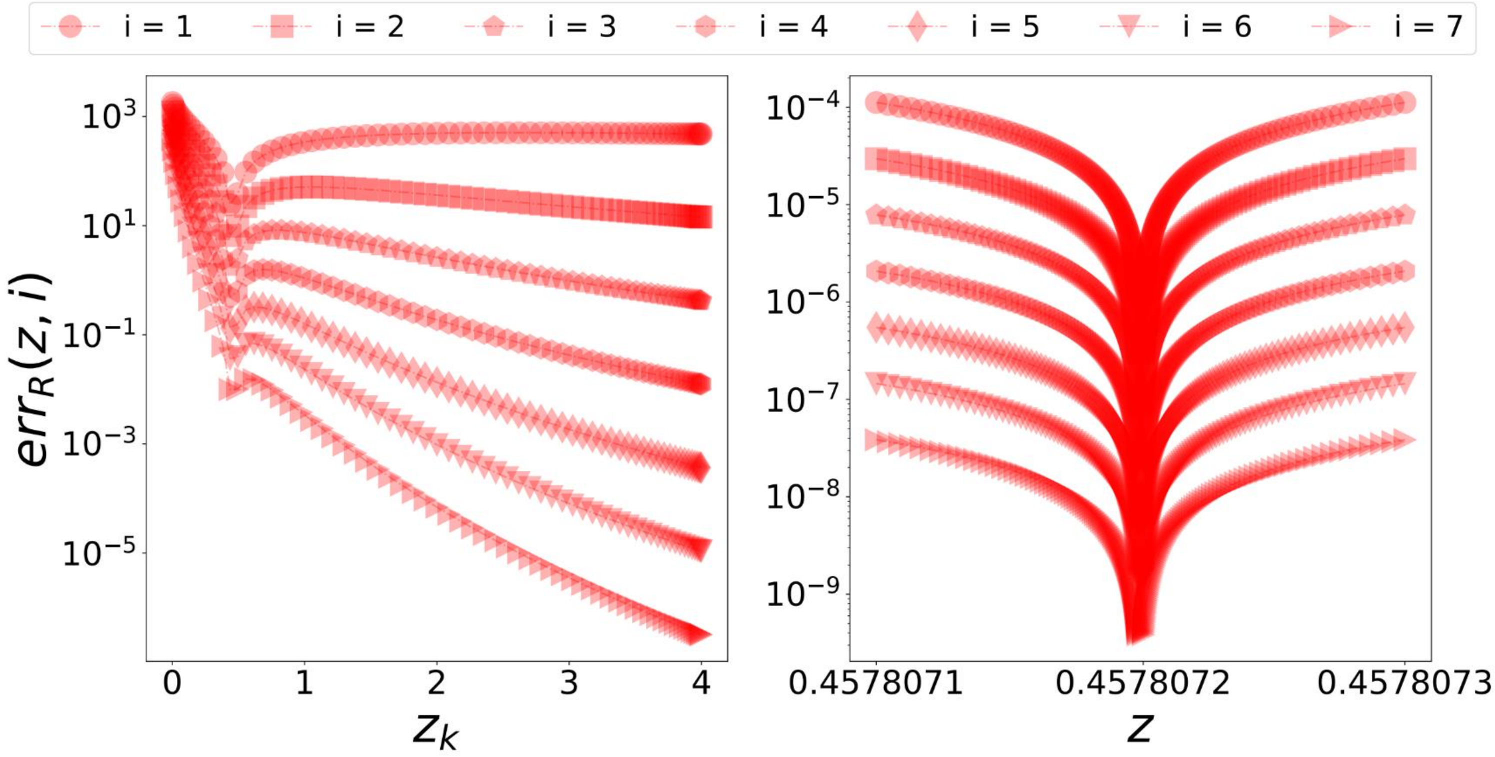
$$\bar{D}_{N^b} := \frac{1}{2}\left(D_{N^b} + (2ph)I_N\right)$$

Improving ABC – Robin



Improving ABC – Pade

Improving ABC – Padé

Having $\lambda = 2 + z$ we get

Theorem. *The function* $\hat{t}^{b}_{N^a}(z)$ *is the* $[\,i\,,\,i\,]$*-Padé approximation about the expansion point* $z = +\infty$ *of* $\hat{t}^{\infty}_{N^a}(z)$*, where* $i = N^b - N^a$.

Improving ABC – Padé

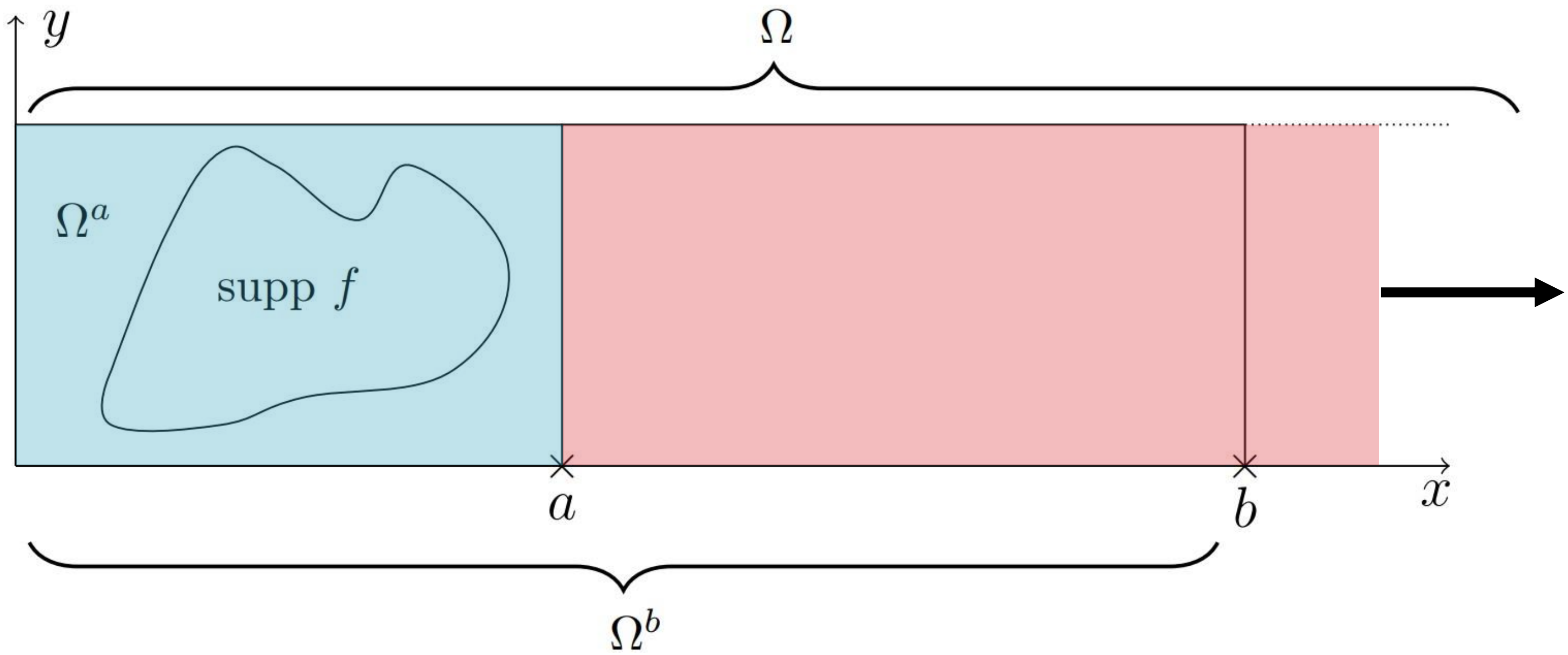
Having $\lambda = 2 + z$ we get

Theorem. *The function* $\hat{t}^{b}_{N^a}(z)$ *is the* $[\,i\,,\,i\,]$*-Padé approximation about the expansion point* $z = z_0$ *of* $\hat{t}^{\infty}_{N^a}(z)$*, where* $i = N^b - N^a$.

Improving ABC – Padé

$$\begin{pmatrix} D_1 & -I_N & & & & \\ -I_N & \ddots & \ddots & & & \\ & \ddots & \breve{D}_{N^a} & -J & & \\ & & -M & \breve{D}_{N^a+1}M & \ddots & \\ & & & \ddots & \ddots & -M \\ & & & & -M & \breve{D}_{N^b}M \end{pmatrix}$$

Improving ABC – Padé



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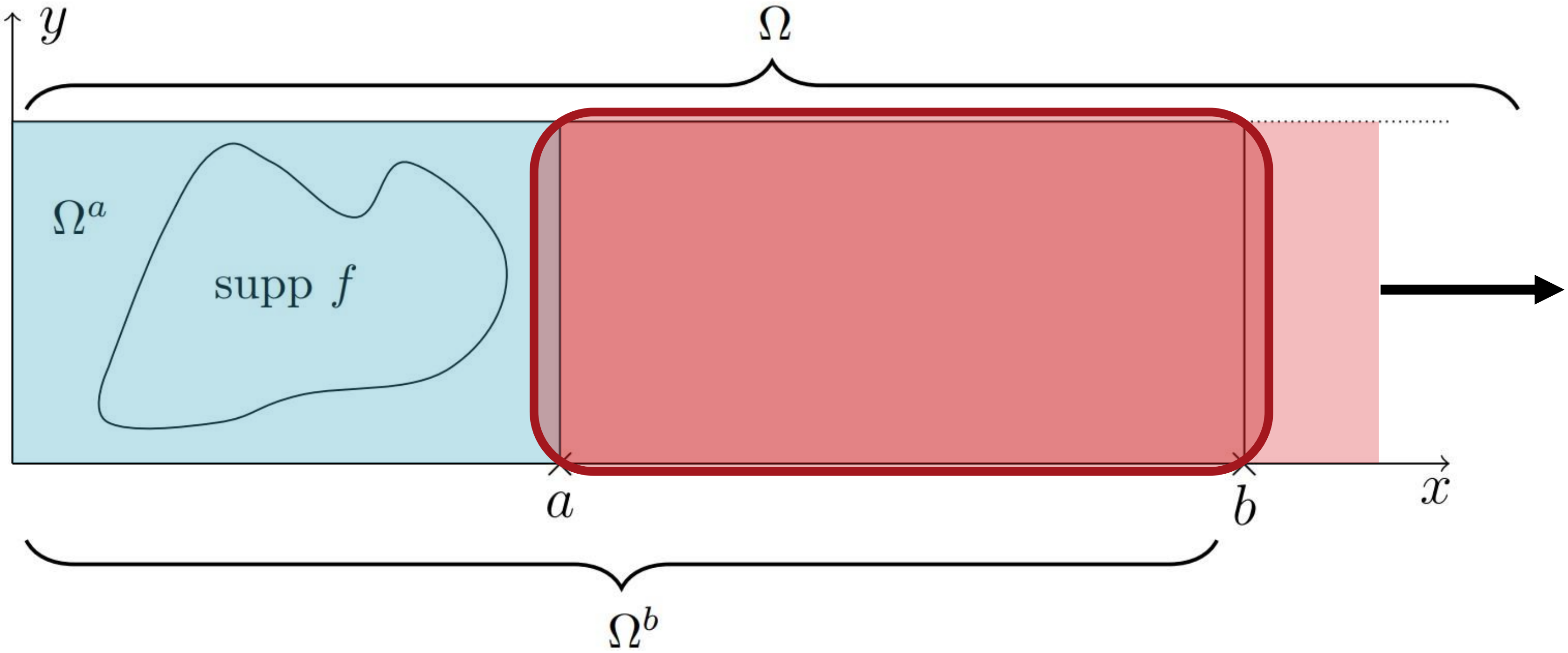

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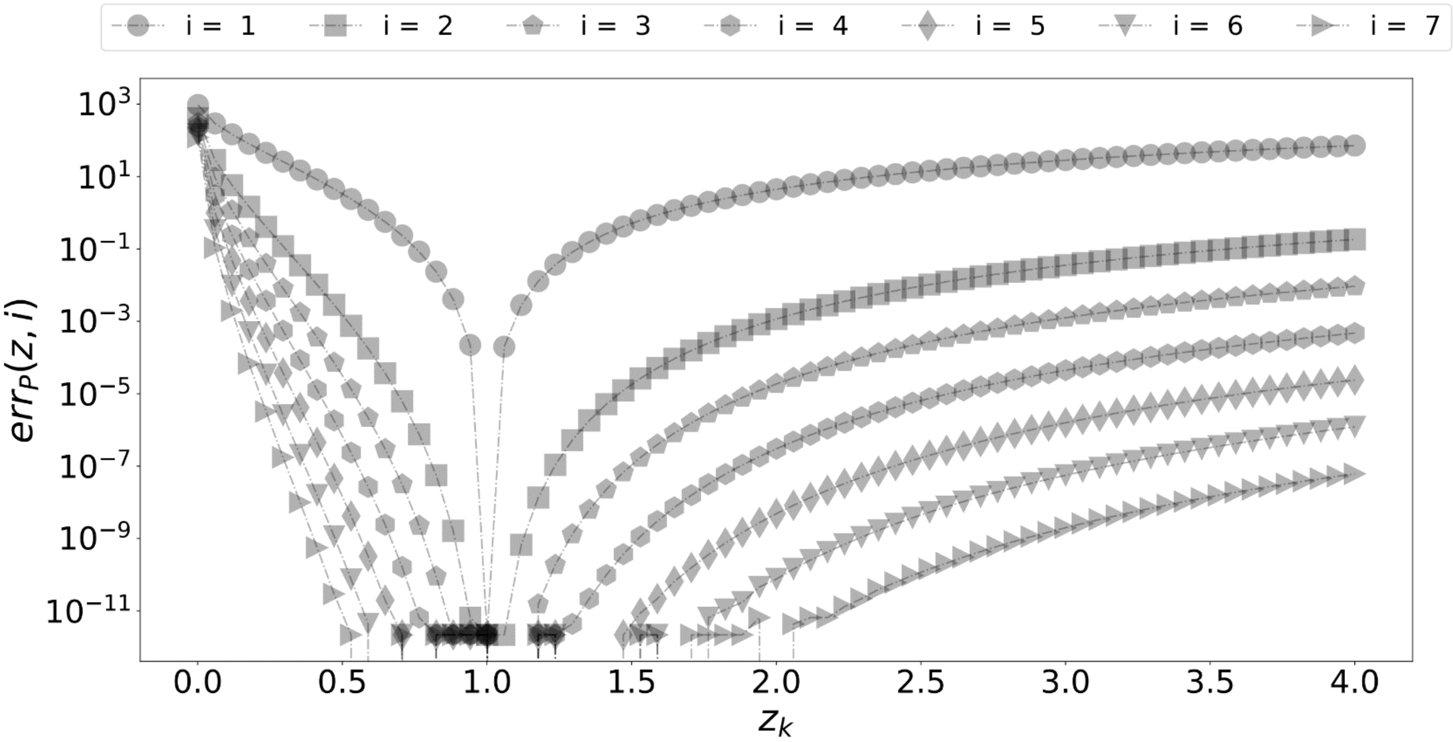

Conclusion

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i	$p^*(i)$	$\frac{\|err_D\|_\infty}{\|err_R\|_\infty}$
1	27.4013	2.569
2	13.7783	3.924
4	8.2295	5.167
8	5.6016	6.598
16	4.3271	8.940

Conclusion

i	$p^*(i)$	$\frac{\lVert err_D \rVert_\infty}{\lVert err_R \rVert_\infty}$
1	27.4013	2.569
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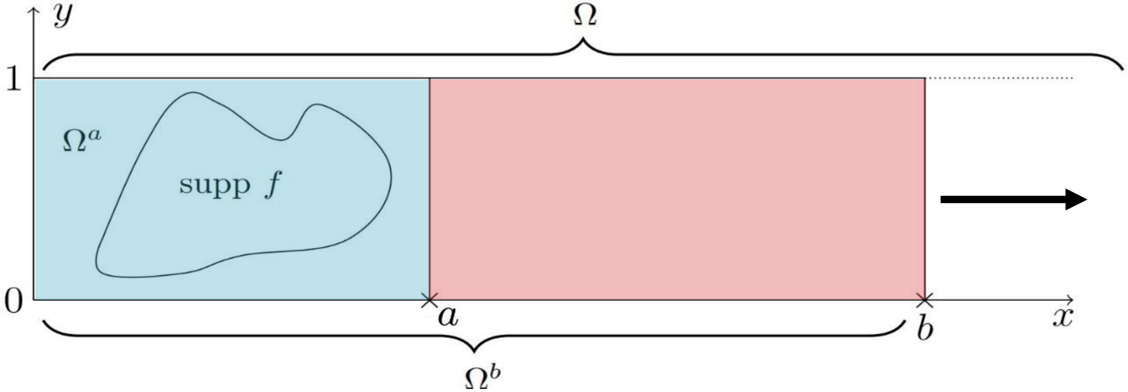
16	4.3271	8.940

i	optimal z_0	$\frac{\lVert err_D \rVert_\infty}{\lVert err_P \rVert_\infty}$	$\frac{\lVert err_R \rVert_\infty}{\lVert err_P \rVert_\infty}$
1	0.4356	3.691	1.441
2	0.2101	10.091	2.572
4	0.1409	18.446	3.569
8	0.0932	86.163	13.058
16	0.0680	3595.822	402.186

Thank you for your attention

Schwarz methods & ABC



$$L^a \mathbf{u}^a = \mathbf{f}^a$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix}$$

$$L^b \mathbf{u}^b = \mathbf{f}^b$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & T^b_{N^a} \end{pmatrix}$$

$$L \mathbf{u} = \mathbf{f}$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & T^\infty_{N^a} \end{pmatrix}$$